The Weak-Heap Family of Priority Queues in Theory and Praxis

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These slides are available at http://www.cphstl.dk

Our research program

Provide the best bounds on the comparison complexity of priority-queue operations (n the current size of the data structure):

find-min: no element comparisons
delete, *delete-min*: $\sim\beta \lg n$ element comparisons
Other operations: $\sim\kappa$ element comparisons
where β and κ are constants.

worst case per operation

Full operation repertoire: *insert*, *borrow*, *decrease*, *meld*

New game

Application = Request sequence
(Think graph applications)

What is the best bound when handling a request sequence consisting of n *insert*, n *delete-min*, and m *decrease* operations?

worst case per sequence

Our result

Rank-relaxed weak heaps are **better** than Fibonacci heaps!

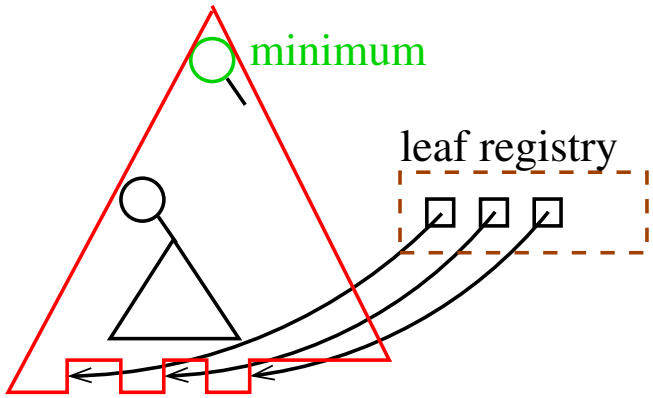
Data structure	**# element comparisons**
Fibonacci heap	$2m + 2.89n \lg n$
Rank-relaxed weak heap	$2m + 1.5n \lg n$

But they are not **simpler**!

Data structure	**Lines of code**
Binary heap	205
Fibonacci heap	296
Rank-relaxed weak heap	883

Weak heap

- A binary tree
- The root only has a right child if any
- Elements obey the weak-heap ordering: for element x, every element in the right subtree is $\geq x$
- With the exception of the root, the nodes that have at most one child are at the last two levels only



Our representation pointer-based!

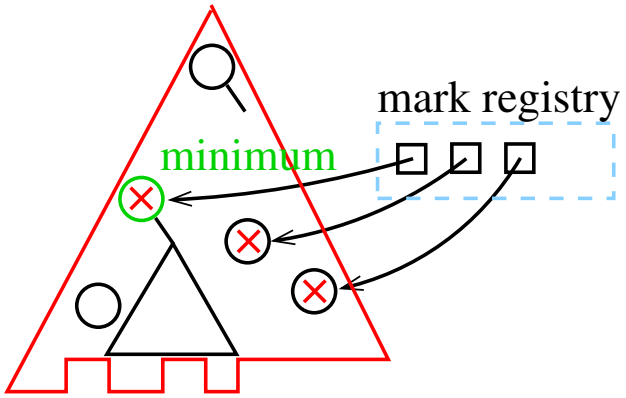
Rank-relaxed weak heap

- $\lambda \leq \lfloor \lg n \rfloor - 1$ nodes marked; they may violate the weak-heap ordering

insert: Insert a leaf, mark it, apply λ-reducing transformations as long as possible.

decrease: Decrease the value in the given node, mark it, apply λ-reducing transformations as long as possible.

delete-min: Find the minimum (at the root or one of the marked nodes), borrow a leaf, fix the structure of the subtree that lost its root, mark the root of the fixed subtree, apply λ-reducing transformations as long as possible.



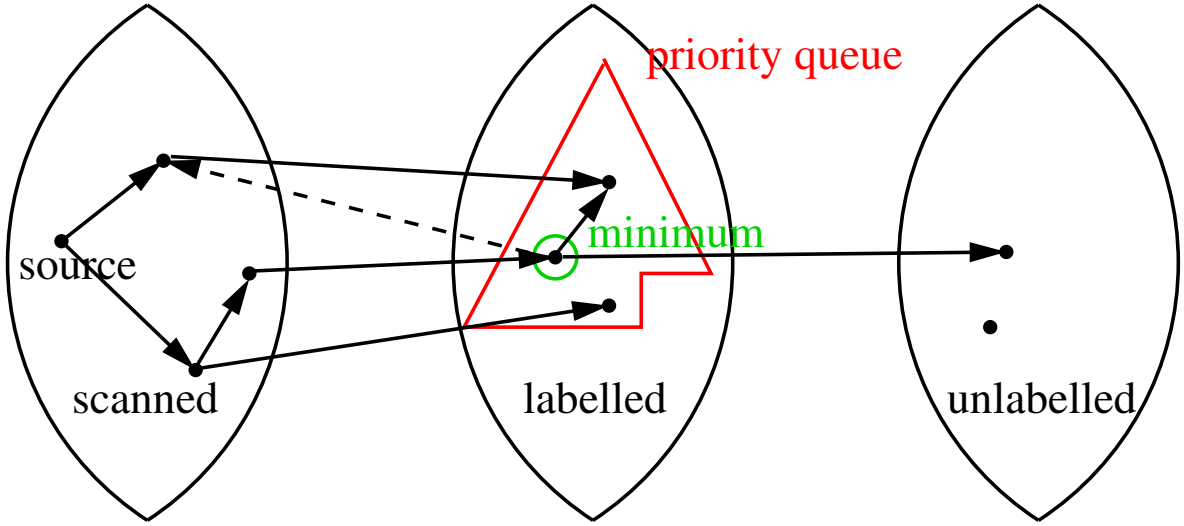
Improvement in *delete-min*: If the mark registry is more than half full before the minimum finding, empty it.

Normal referee comments

- Element comparisons are not relevant in practice!
- Pairing heaps are much faster in practice!
- No one would ever implement Fibonacci heaps since they are so complicated.

This is you guys! Thank you! You pushed us to do some practical experiments!

Our play with Dijkstra's algorithm



a factor of two speed-up

With your search engine, you will find many experimental studies on Dijkstra's algorithm. Be critical when you read the results.

- Which algorithm
- Which graph representation
- Which priority queue
- Which tuning level

Graph representation

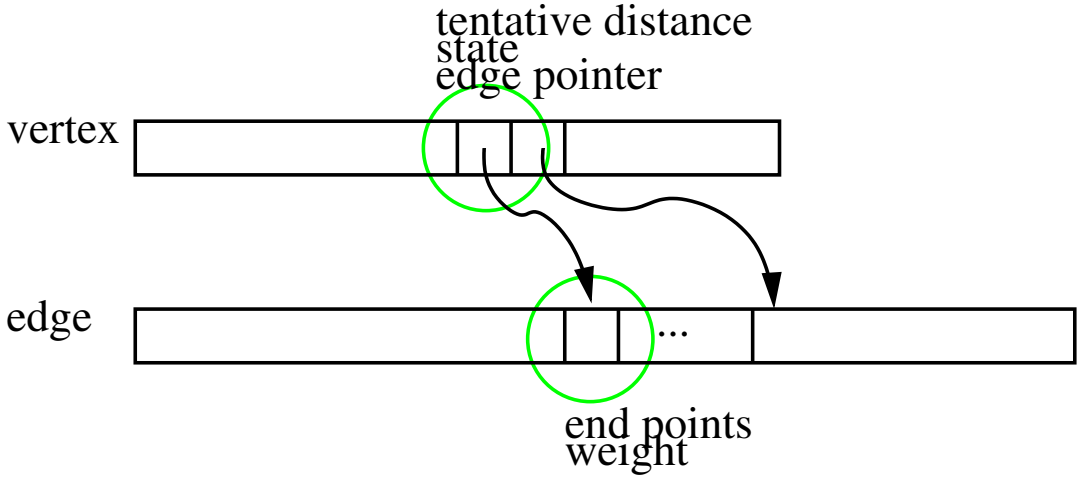
CPH STL

- adjacency arrays
- simple
- static
- $16m + 16n + O(1)$ bytes for a graph with m edges and n vertices

LEDA

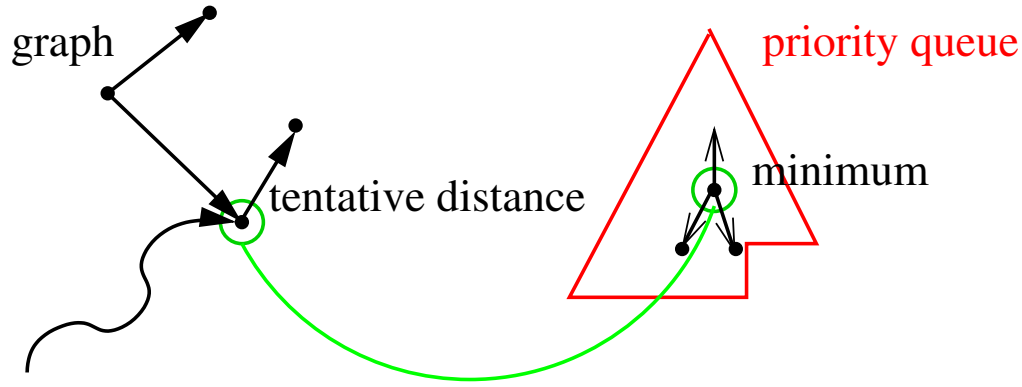
- adjacency lists
- nice interface
- fully dynamic
- parameterized
- $52m + 60n + O(1)$ bytes [LEDA Book, § 6.14]

weights doubles



a factor of two speed-up

Interaction



Combine the graph vertex and the priority-queue node [Knuth 1994]

$\rightarrow$ improves cache behaviour

a factor of two speed-up

Tuning

Running time per n [μs]

Structure / Operation	CPH STL Fibonacci heap	LEDA 6.2 Fibonacci heap
insert		
n: 10 000	0.10	0.18
n: 100 000	0.09	0.15
n: 1 000 000	0.09	0.15
decrease		
n: 10 000	0.03	0.06
n: 100 000	0.05	0.22
n: 1 000 000	0.06	0.31
delete-min		
n: 10 000	0.7	1.2
n: 100 000	1.4	2.7
n: 1 000 000	2.8	4.5

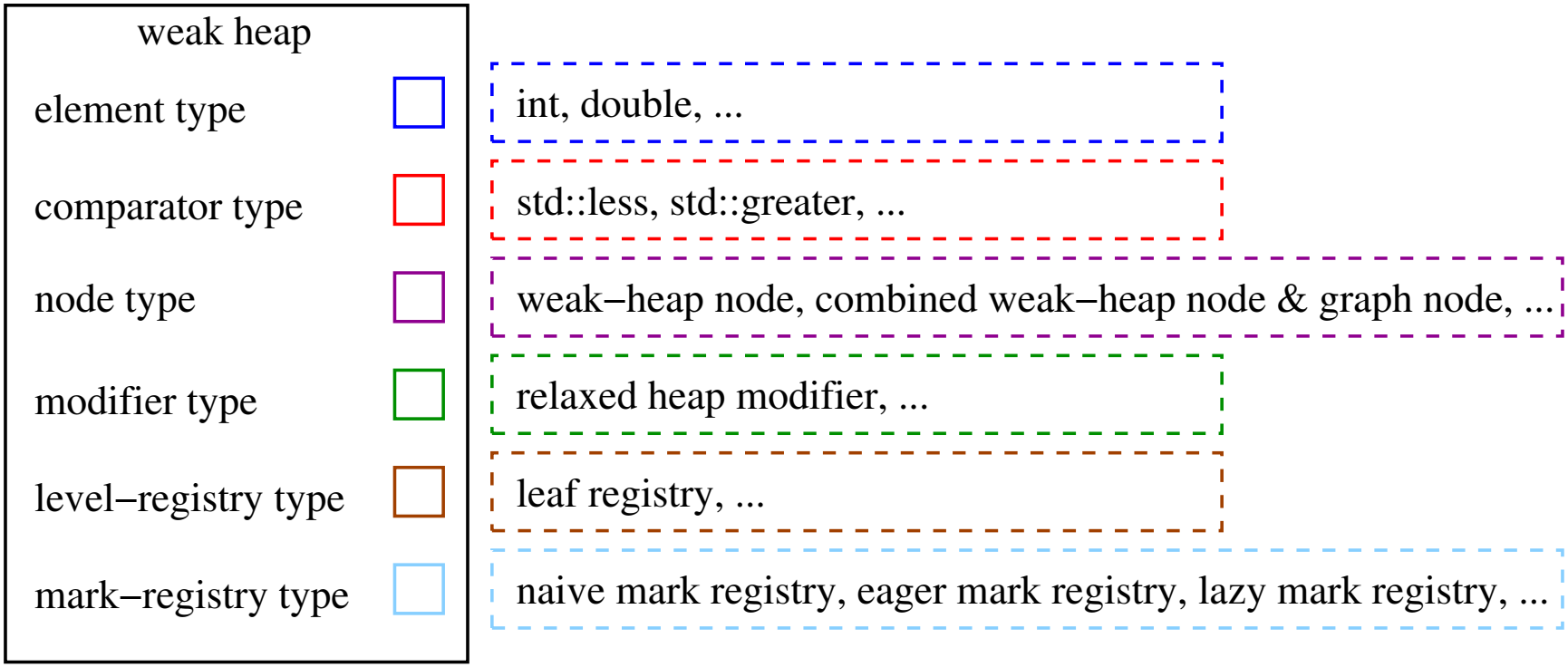
Element comparisons per n

Structure / Operation	CPH STL Fibonacci heap	LEDA 6.2 Fibonacci heap
insert		
n: 10 000	0	1
n: 100 000	0	1
n: 1 000 000	0	1
decrease		
n: 10 000	0	2
n: 100 000	0	2
n: 1 000 000	0	2
delete-min		
n: 10 000	16.2	29.9
n: 100 000	21.2	38.3
n: 1 000 000	26.2	46.5

a factor of two speed-up

On my computer (Ubuntu, gcc, with -O3)

Parameterized design



- comparators shared
- nodes shared
- transformations shared
- level registries shared
- mark registries shared

a factor of two less code

What is the best?

Our reference sequence

Theory: rank-relaxed weak heap
Dijkstra—time: binary heap
[Williams 1964]
Dijkstra—comps: weak heap
[Dutton 1993]

Worst case per operation

insert—**time:** Fibonacci heap
[Fredman & Tarjan 1987]
insert—**comps:** Fibonacci heap
decrease—**time:** Fibonacci heap
decrease—**comps:** Fibonacci heap
delete-min—**time:** weak queue
[Vuillemin 1978]
delete-min—**comps:** weak heap

Conclusions

Earlier conjectures—see my SWAT-2010 slides

Katajainen's third conjecture: Our reference sequence can be handled with $2m + n \lg n + o(n)$ element comparisons in $O(m + n \lg n)$ time.

Cache effects: We have neglected the cache behaviour of priority queues for too long.

Source code: Our programs are available via the CPH STL website.