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Title:

Relaxed weak queues: an alternative to run-relaxed heaps

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These slides as well as the underlying paper are available at <http://www.cphstl.dk/>.

Priority-Queue Operations

insert

input: element
output: locator

find-min

input: none
output: locator

delete-min

$p \leftarrow \text{find-min}()$
 $\text{delete}(p)$

delete

input: locator
output: none

decrease

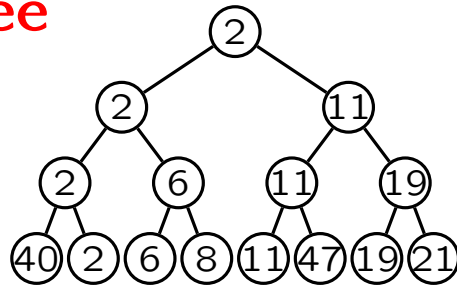
input: locator, element
output: none

meld

input: two priority queues
output: one priority queue

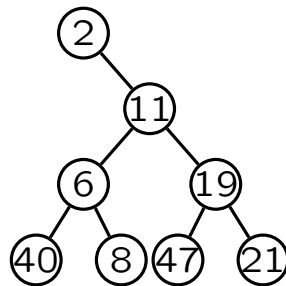
Various Approaches

winner tree



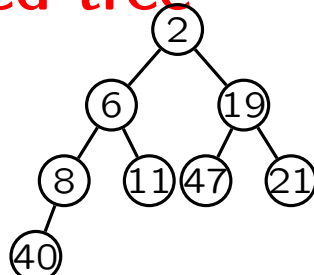
selection tree
navigation pile
binomial tree

loser tree



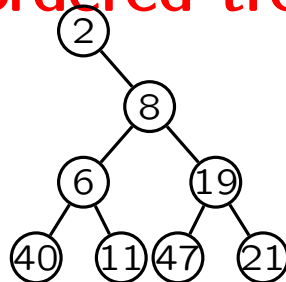
Vheap

heap-ordered tree



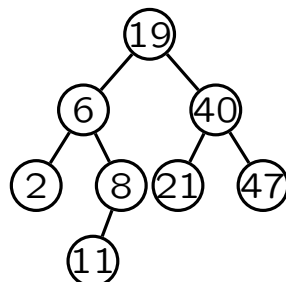
binary heap
leftist heap

weak-heap-ordered tree



weak heap

search tree



AVL tree

⋮

Market Analysis

<div style="display: inline-block; transform: rotate(-45deg);"> efficiency method </div>	binary heap worst case	binomial queue worst case	Fibonacci heap amortized	run-relaxed heap worst case
<i>find-min</i>	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
<i>insert</i>	$\Theta(\lg n)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
<i>decrease</i>	$\Theta(\lg n)$	$\Theta(\lg n)$	$\Theta(1)$	$\Theta(1)$
<i>delete</i>	$\Theta(\lg n)$	$\Theta(\lg n)$	$\Theta(\lg n)$	$\Theta(\lg n)$
<i>meld</i>	$\Theta(\lg m \times \lg n)$	$\Theta(\min\{\lg m, \lg n\})$	$\Theta(1)$	$\Theta(\min\{\lg m, \lg n\})$

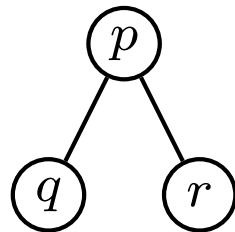
Here m and n denote the number of elements in the priority queues just prior to the operation.

Our Work

Relaxed weak queues — an alternative to run-relaxed heaps:

- are simpler to program,
- work on a pointer machine except that *meld* requires random access [pointer machine $\approx C$ without arrays],
- are asymptotically equally fast,
- have low constant factors [*delete* requires $3 \lg n + O(1)$ element comparisons; can be improved to $\lg n + O(\lg \lg n)$], and
- use less space [$3n + O(\lg n)$ extra words; $4n + O(\lg n)$ with *meld*].

Nonstandard Tree Terminology



- p is the **surrogate parent** of q .
- p is the **real parent** of r .
- Let s be a node in a binary tree. We call every ancestor of s that is a real parent of another ancestor of s a **real ancestor** of s .

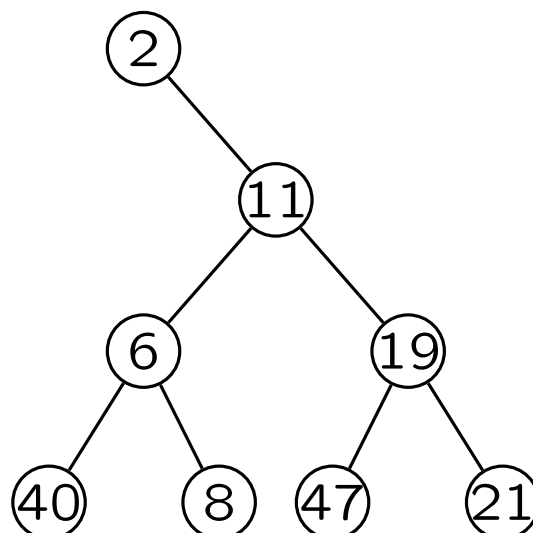
Perfect Weak Heaps

A **perfect weak heap** is a binary tree having the following three properties:

1. The root has no left subtree.
2. The right subtree of the root is a complete binary tree.
3. For every node s , the element stored at s is no smaller than the element stored at the **first** real ancestor of s .

Fact 1. A perfect weak heap stores 2^h elements for some integer $h \geq 0$.

Fact 2. The root of a perfect weak heap must store a minimum element.



Weak Queues

A **weak queue** Q storing n elements is a collection of disjoint perfect weak heaps. Consider the binary representation of n

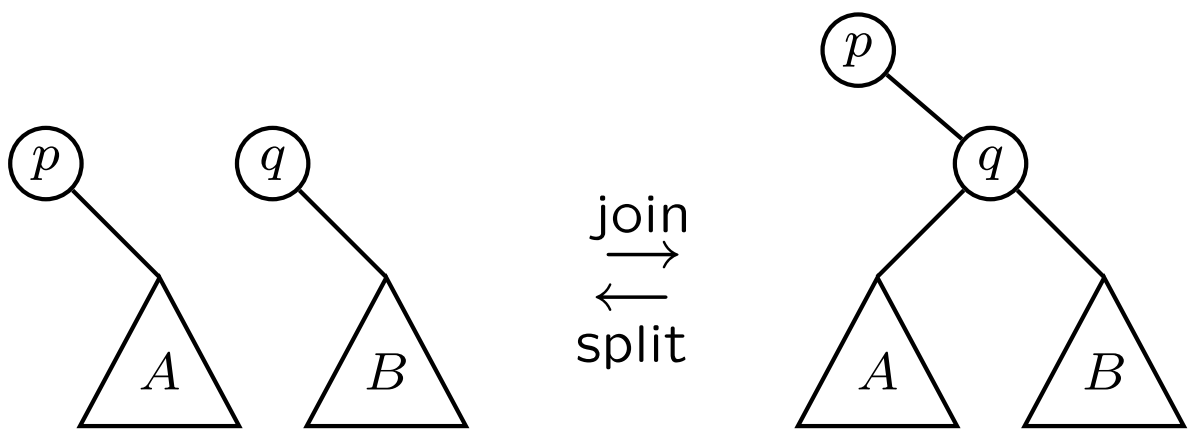
$$n = \sum_{i=0}^{\lfloor \lg n \rfloor} b_i 2^i,$$

where $b_i \in \{0, 1\}$ for all $i \in \{0, \dots, \lfloor \lg n \rfloor\}$. In its basic form, Q contains a perfect weak heap H_i of size 2^i if and only if $b_i = 1$, i.e.

$$Q = \{H_i \mid n = \sum_{i=0}^{\lfloor \lg n \rfloor} b_i 2^i \text{ and } b_i = 1\}.$$

Primitive Operations

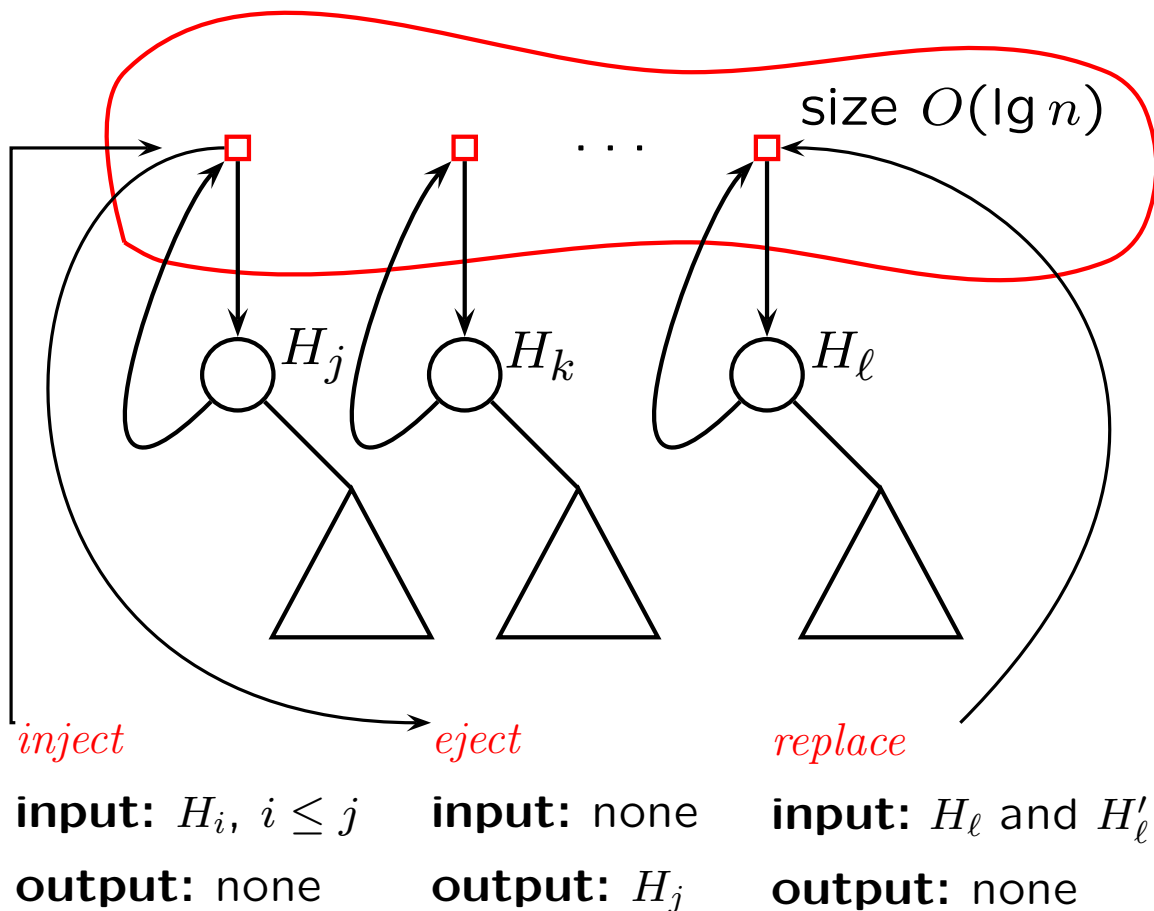
Joining and splitting two perfect weak heaps of the same size:



Note that for a binary heap a join may take logarithmic time.

Heap Store

A **heap store** is a sequence of perfect weak heaps appearing in increasing order of height.



Idea. Injections are done lazily by not doing all joins at once; we allow between zero and two perfect weak heaps of each size.

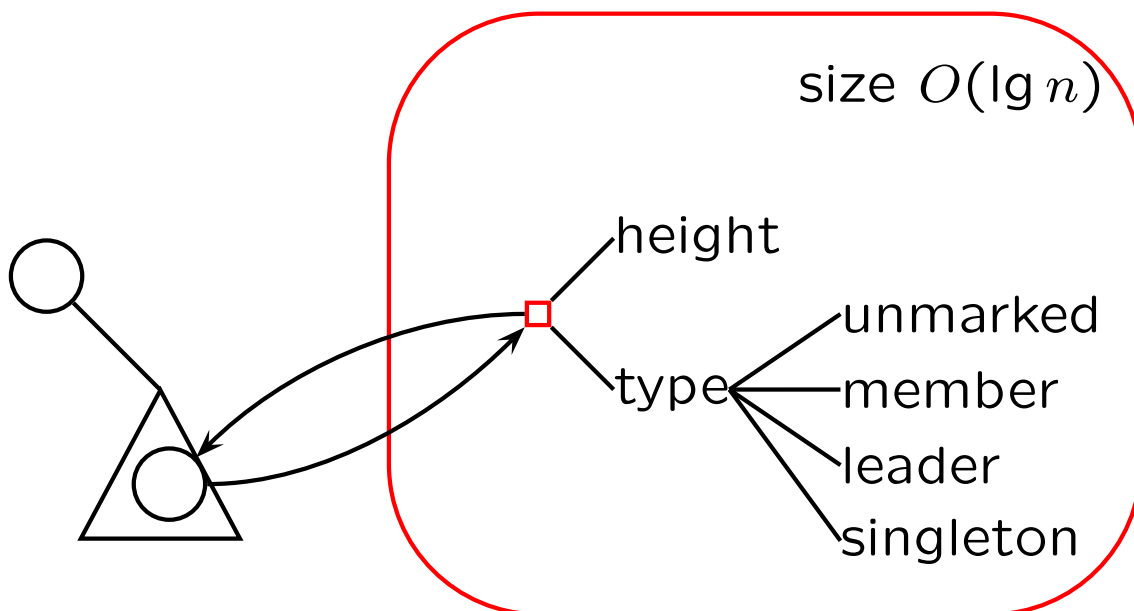
Theorem 1. All heap-store operations *inject*, *eject*, and *replace* take $O(1)$ worst-case time.

Potential Violation Nodes

- A **weak-heap-order violation** occurs if the element stored at a node is smaller than the element stored at the **first** real ancestor of that node. In a **marked node** a weak-heap-order violation may occur.
- A marked node is **tough** if it is the left child of its parent and also the parent is marked.
- A chain of consecutive tough nodes followed by a single nontough marked node is called a **run**.
- All tough nodes of a run are called its **members**.
- The single nontough marked node of a run is called its **leader**.
- A marked node that is neither a member nor a leader of a run is called a **singleton**.

Node Store

The primary purpose of a **node store** is to keep track of potential violation nodes, and its secondary purpose is to store the heights and types of the nodes.



mark

input: a node

output: none

unmark

input: a node

output: none

reduce

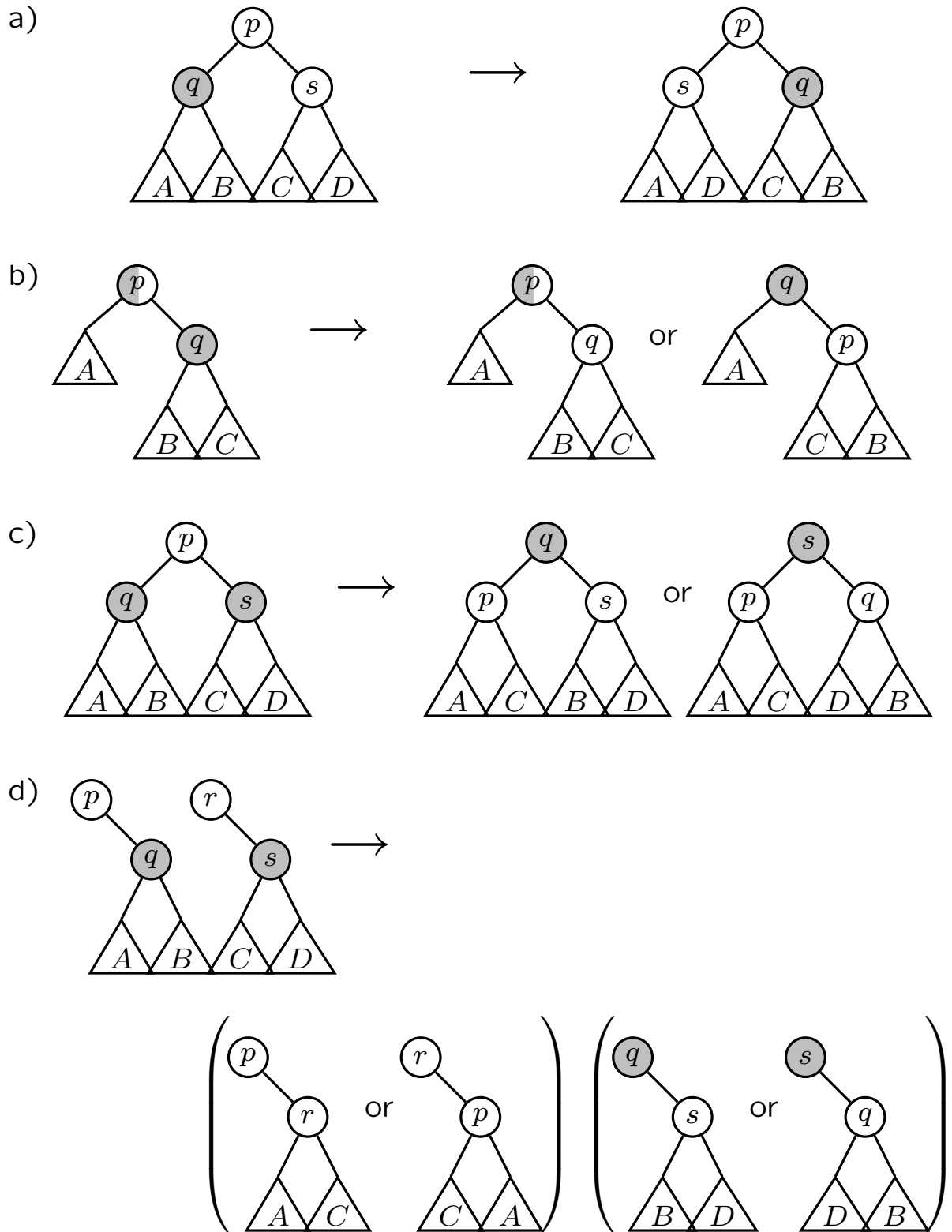
input: none

output: none

effect: Unmark at least one arbitrary marked node.

Theorem 2. *The node-store operations mark, unmark, and reduce take $O(1)$ worst-case time.*

Primitives Used by *reduce*



find-min()

To facilitate a fast *find-min*, a pointer to the node storing the current minimum is maintained and updated by all modifying operations. This **minimum pointer** refers to a root or to a potential violation node.

The minimum pointer points to the node storing the current minimum, so this node can just be returned.

Worst-case time: $\Theta(1)$; no element comparisons

insert(e)

1. Allocate a new node and put e there.
2. Place the new node, which is also a perfect weak heap of height 0, into the heap store by invoking *inject*.
3. Correct the minimum pointer to point to the new node if e is smaller than the current minimum.

Worst-case time: $\Theta(1)$ with at most 2 element comparisons

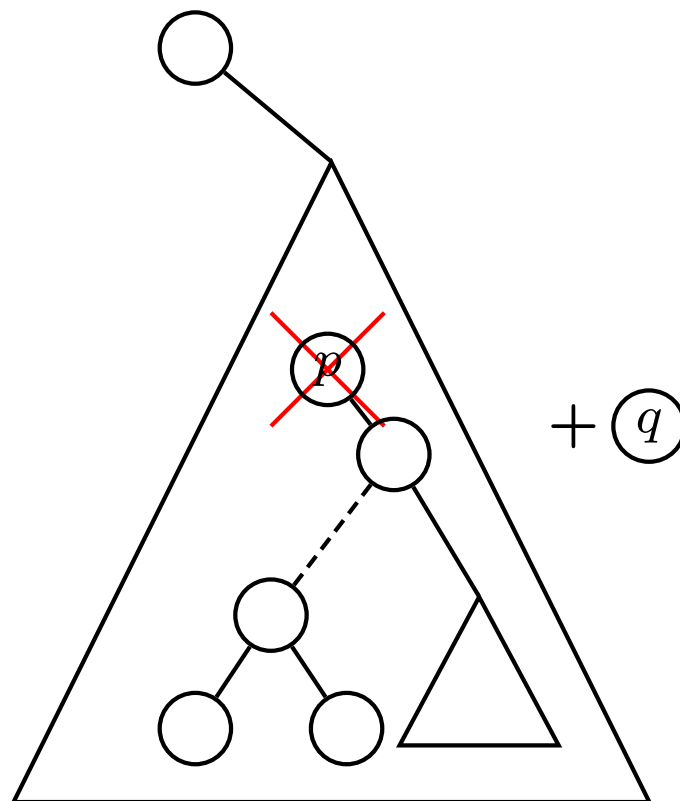
decrease(p, e)

1. Make the element replacement at p .
2. Make p a potential violation node by invoking *mark*.
3. Reduce the number of potential violation nodes, if possible, by invoking *reduce*.
4. Correct the minimum pointer if necessary.

Worst-case time: $\Theta(1)$ with at most 4 element comparisons

delete(p)

The idea is to extract the subheap rooted at p from the perfect weak heap, in which it resides, borrow another node q from the smallest perfect weak heap to fill in the hole created by p , and put the new subheap in the place of the extracted subheap.



Worst-case time: $\Theta(\lg n)$ with at most $3 \lg n + O(1)$ element comparisons

delete(p) — Details

1. Eject the smallest perfect weak heap from the heap store by invoking *eject*. Let q be the root of that perfect weak heap.
2. Repeat until q has no children:
 - a) Split the perfect weak heap rooted at q . Let r be the root of the other subheap created.
 - b) Remove the marking of r , if any, by invoking *unmark*.
 - c) Insert the subheap rooted at r into the heap store by invoking *inject*.
3. If p and q are the same node, go to 11.
4. Extract the subheap rooted at p from the perfect weak heap, in which it resides, and remember its neighbouring nodes.
5. Repeat until p has no children:
 - a) Split the subheap rooted at p . Let s be the root of the other subheap created.
 - b) Push the subheap rooted at s onto a temporary stack.

6. Repeat until the temporary stack is empty:
 - a) Pop the top of the stack. Let s be the root of the subheap popped.
 - b) Remove the marking of s , if any, by invoking *unmark*.
 - c) Join the subheaps rooted at q and s ; independent of the outcome denote the new root q .
7. Put q in the place of p .
8. Make q a potential violation node by invoking *mark*.
9. If p was a root, substitute the perfect weak heap rooted at q for that rooted at p in the heap store by invoking *replace*.
10. Remove the marking of p , if any, by invoking *unmark* to update the node store.
11. If the minimum pointer points to p , scan all roots and all potential violation nodes to find a new minimum element and update the minimum pointer.

12. Reduce the number of potential violation nodes, if possible, by invoking *reduce* twice (once because of the new potential violation node introduced and once more because of the decrement of n).
13. Free p and return.

$meld(Q, R)$

Assume that the sizes of Q and R are m and n , respectively, and that $m \leq n$.

1. Move all singleton and run-leader objects from the node store of Q to the node store of R .
2. Eject all perfect weak heaps of Q and store them to a temporary stack S_Q .
3. Eject all perfect weak heaps of R , whose height is no greater than $\lfloor \lg m \rfloor$, and store them to a temporary stack S_R .
4. Process all perfect weak heaps in S_Q and S_R in height order and inject them into the heap store of R .
5. Reduce the number of potential violation nodes in R by at most $\lfloor \lg m \rfloor$.
6. Destroy Q and return R .

Worst-case time: $\Theta(\lg m)$ with at most $5 \lg m$ element comparisons

Open Problems

- Is fast *meld* possible without random access?
- Can the number of element comparisons performed by *delete* be reduced from $\lg n + O(\lg \lg n)$ to $\lg n + O(1)$?