On the power of structural violations in priority queues

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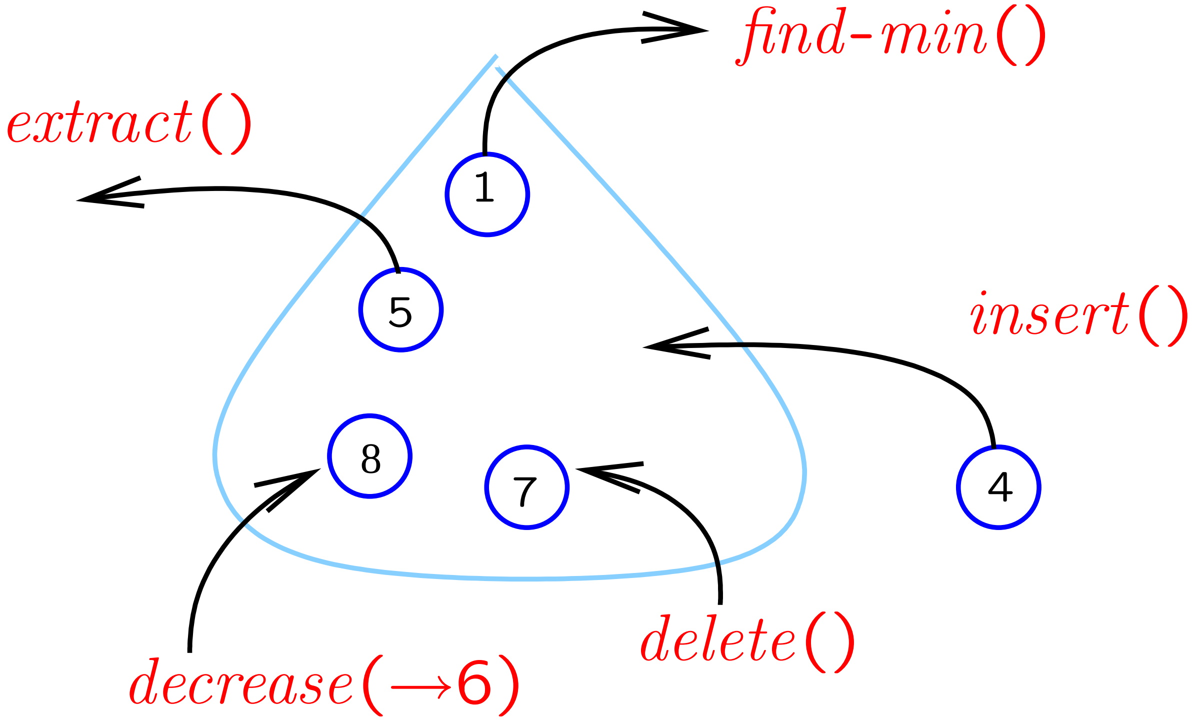
These slides are available at http://www.cphstl.dk

Heaps



Examples:

Fibonacci heaps
Run-relaxed heaps
Fat heaps

Focus

- comparison complexity of heap operations
- worst-case efficiency
- constant factors

Our ultimate goal is to develop a library component that guarantees optimal complexity bounds, but unfortunately the data structures developed are **not** practical.

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Research question

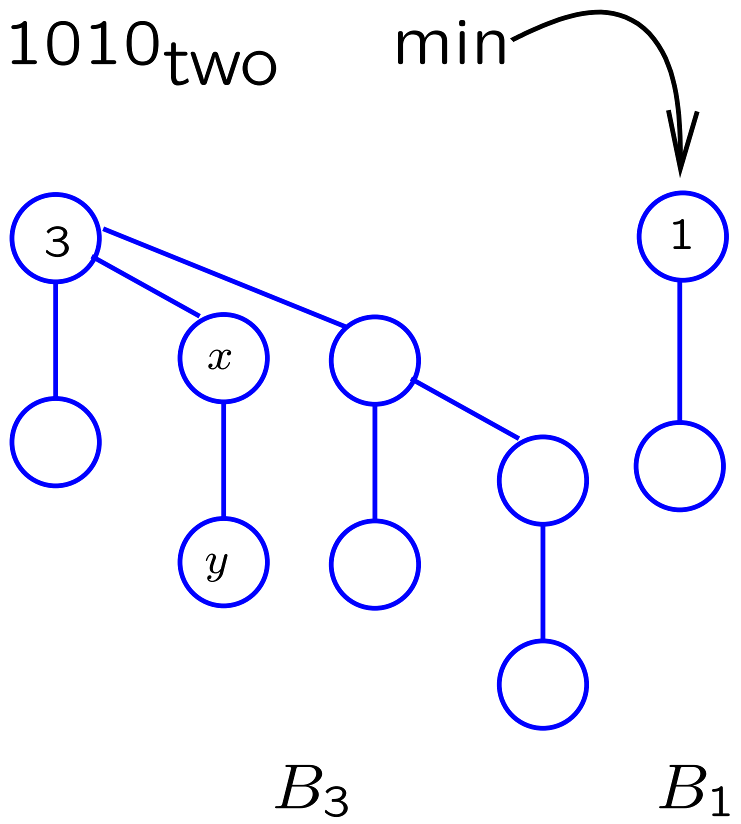
Q: If *find-min*, *insert*, *extract*, and *decrease* are required to take $O(1)$ time, can *delete* be realized in $O(\lg n)$ time including only $\lg n + O(1)$ **element comparisons**.

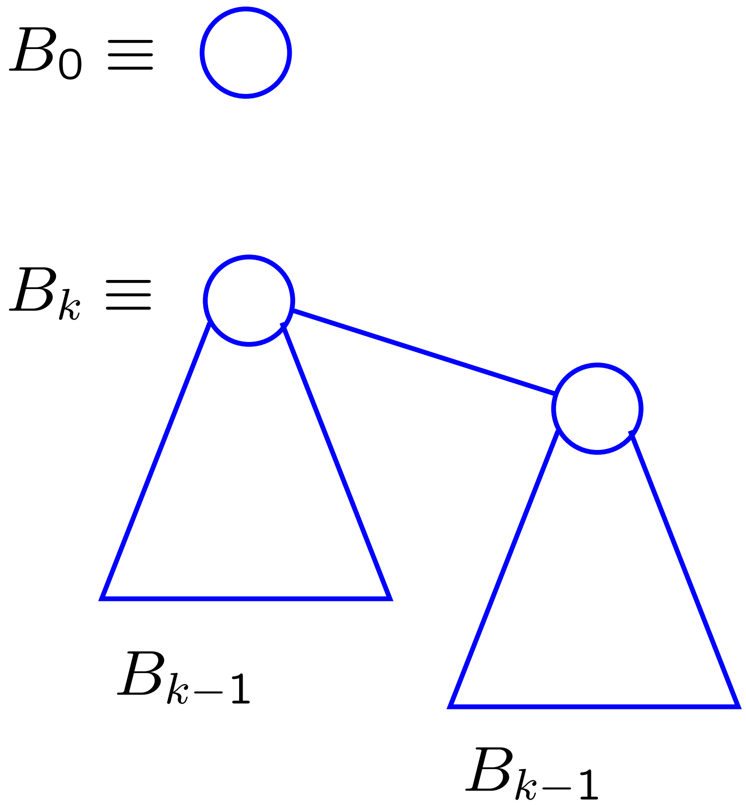
A: If *decrease* is allowed to take $O(\lg n)$ time, **yes**. With $O(1)$-time *decrease* **almost**, but we do not know the final answer.

n: # elements stored just prior to an operation

Binomial heaps

$n = 1010_{\text{two}}$

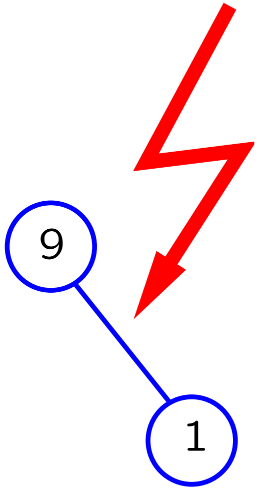




- heap-ordered $x \leq y$
- at most $\lfloor \lg n \rfloor + 1$ binomial trees

Read [Cormen et al. 2001]

Heap-order violations

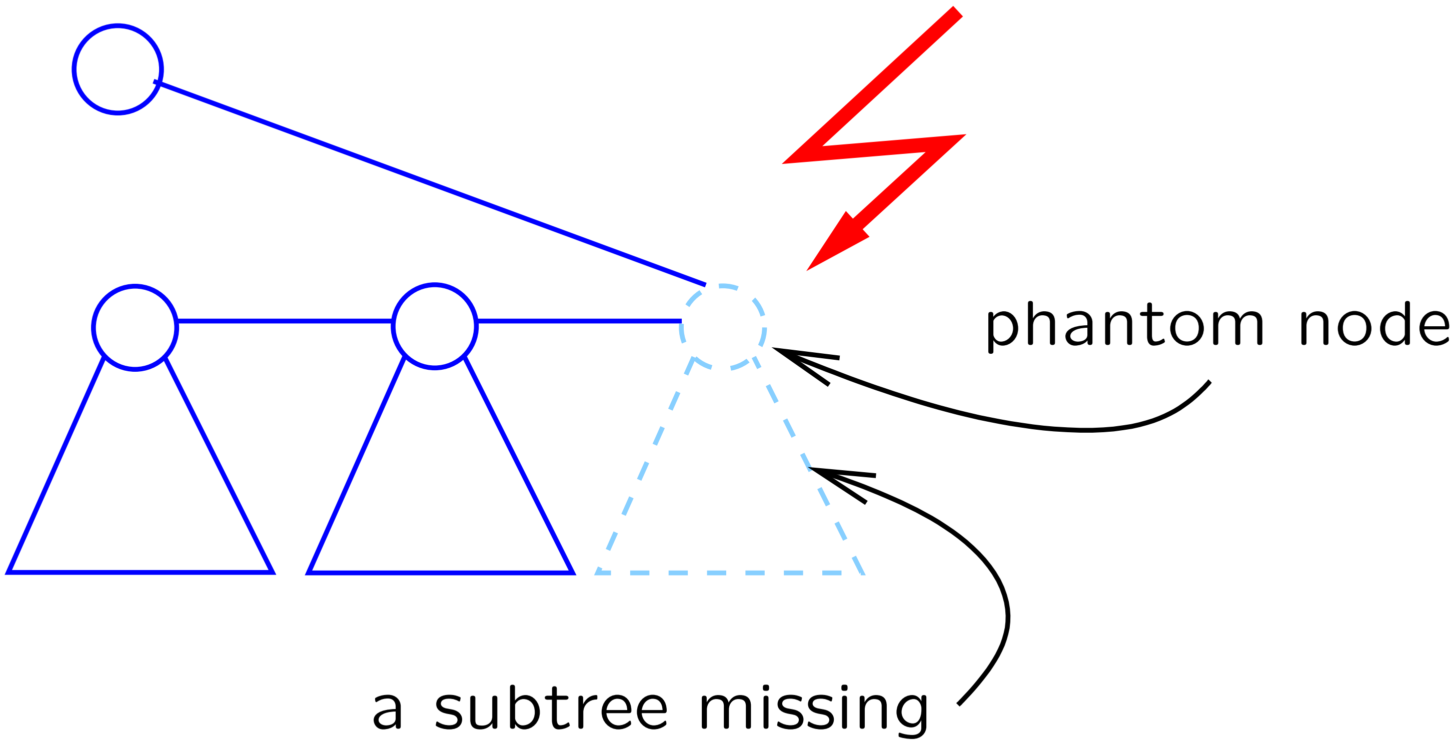


binomial heaps $\Rightarrow$ run-relaxed heaps

$O(\lg n)$ violations

[Driscoll et al. 1988]

Structural violations



Example:

Fibonacci heaps

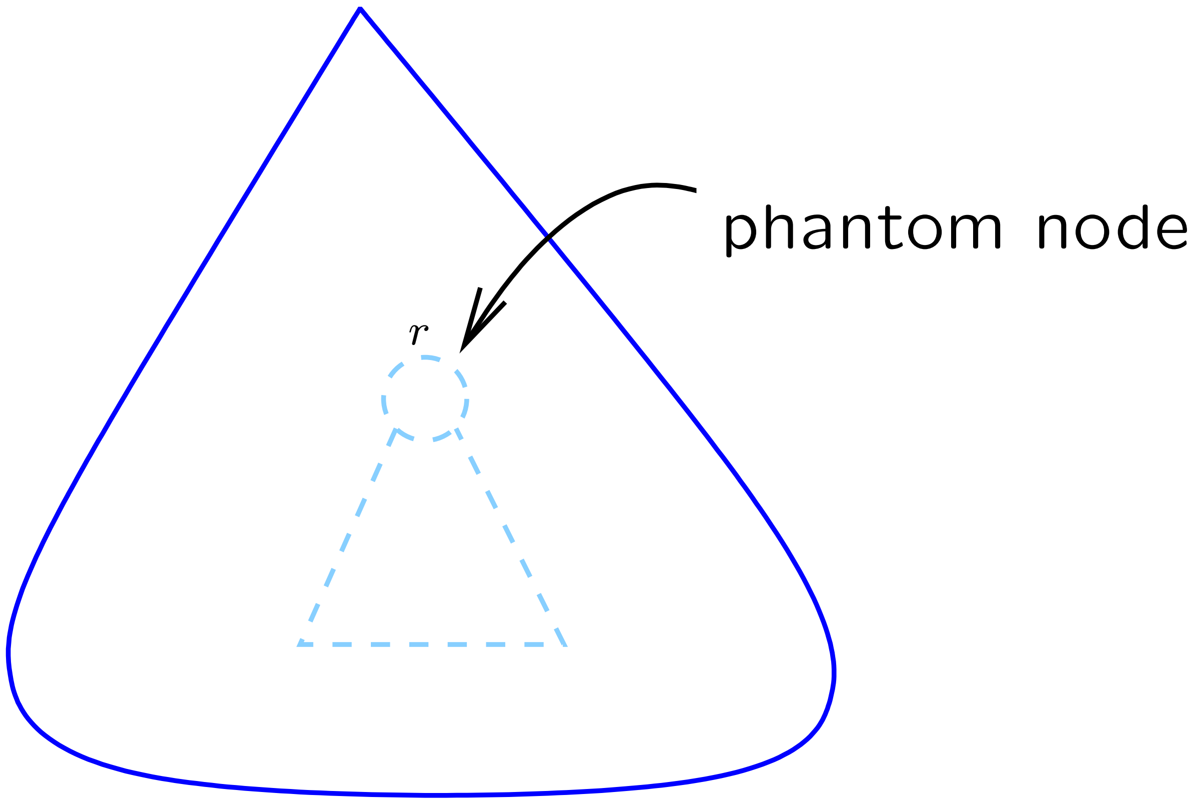
[Fredman & Tarjan 1987]

Pruned heaps

τ: # of trees

λ: # of phantom nodes

$\tau \leq O(\lg n)$
$\lambda \leq O(\lg n)$



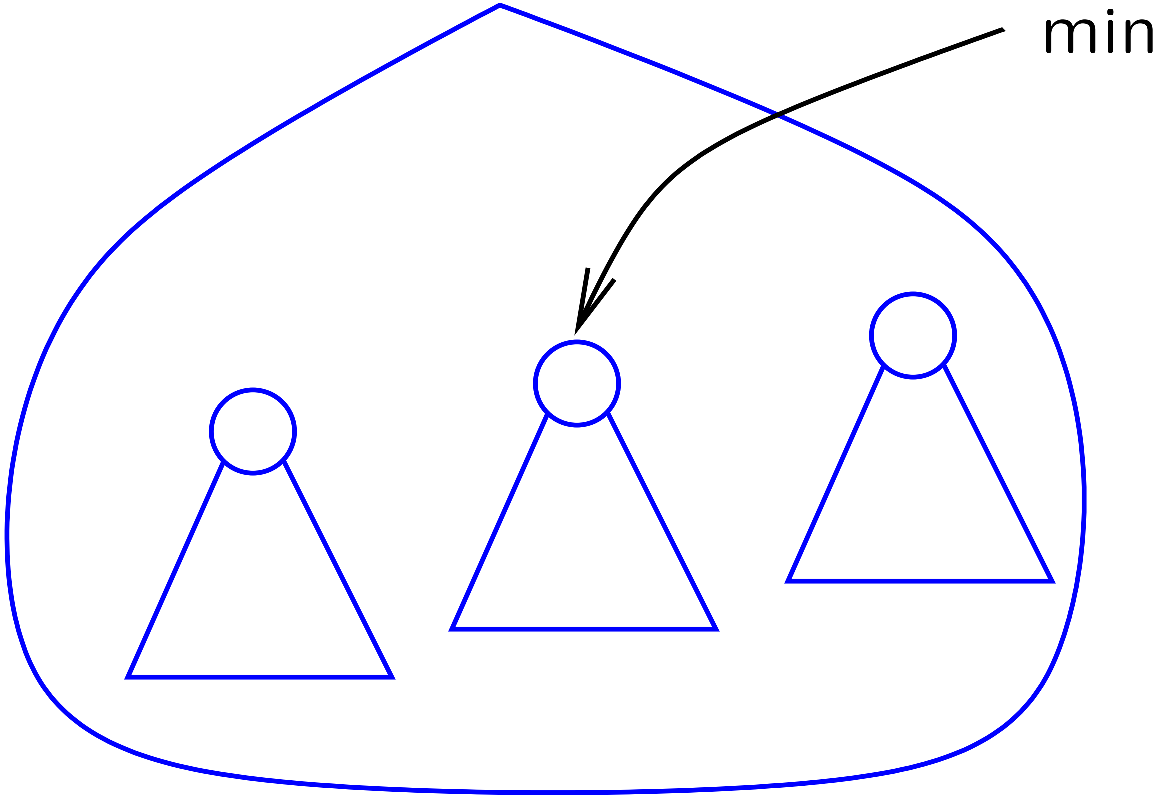
Results

Heap / Operation	Run-relaxed heaps [ISAAC 2006]	Fat heaps	Two-tier relaxed heaps [ISAAC 2006]	Two-tier pruned heaps [this paper]
find-min	$O(1)$	$O(1)$	$O(1)$	$O(1)$
insert	$O(1)$	$O(1)$	$O(1)$	$O(1)$
extract	$O(1)$	–	$O(1)$	$O(1)$
decrease	$O(1)$	$O(1)$	$O(1)$	$O(1)$
delete	$3 \lg n + O(1)$	$2.73 \lg n + O(1)$	$\lg n + O(\lg \lg n)$	$\lg n + O(\sqrt{\lg n})$

# of element comparisons

Find-min

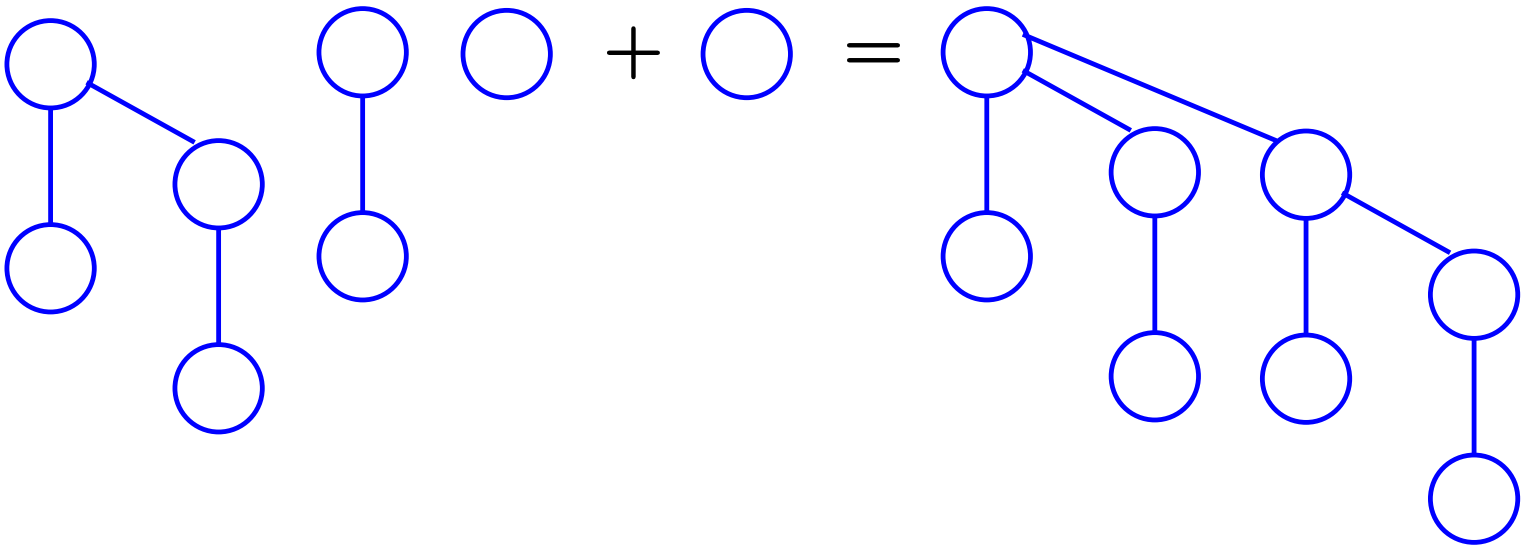
Maintain a pointer to the current minimum



Insert

Imitate ++ for binary numbers

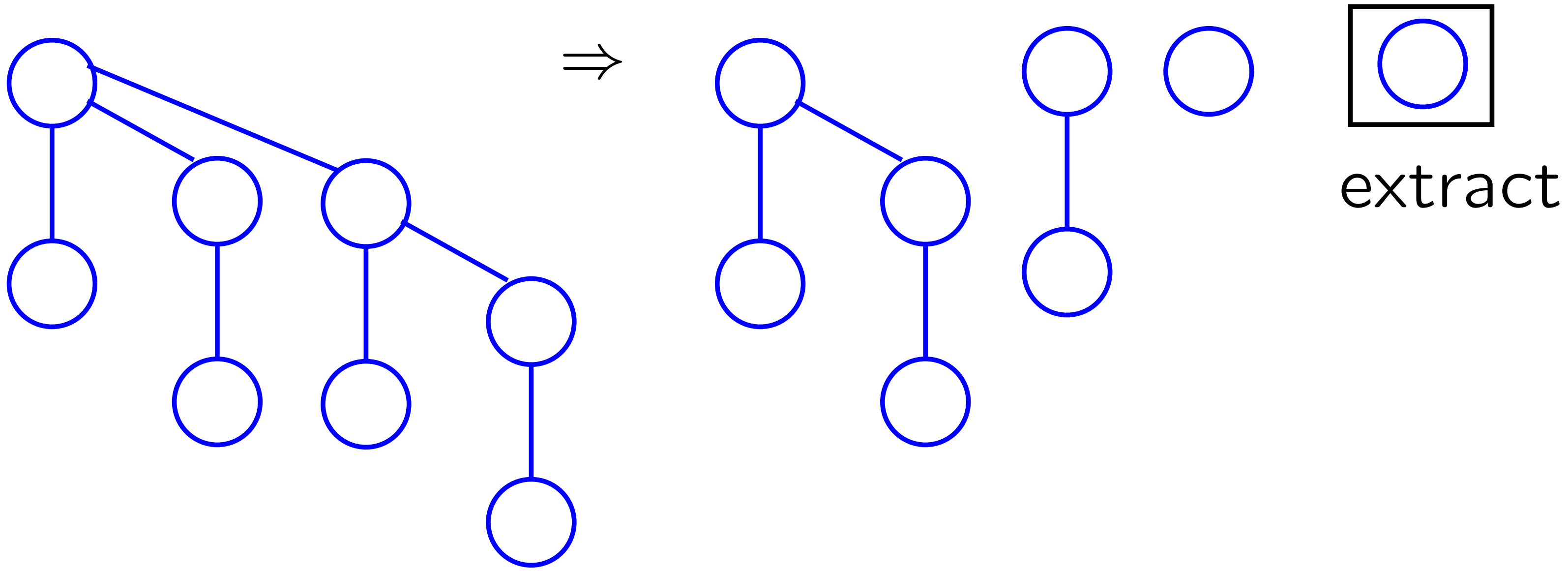
$$\begin{array}{rcccc} & & 1 & 1 & 1 \\ + & & & & 1 \\ \hline & 1 & 0 & 0 & 0 \end{array}$$



Extract

Imitate -- for binary numbers

$$\begin{array}{rrrrr} & 1 & 0 & 0 & 0 \\ - & & & & 1 \\ \hline & & 1 & 1 & 1 \end{array}$$



extract is our contribution for the mankind!

Problem

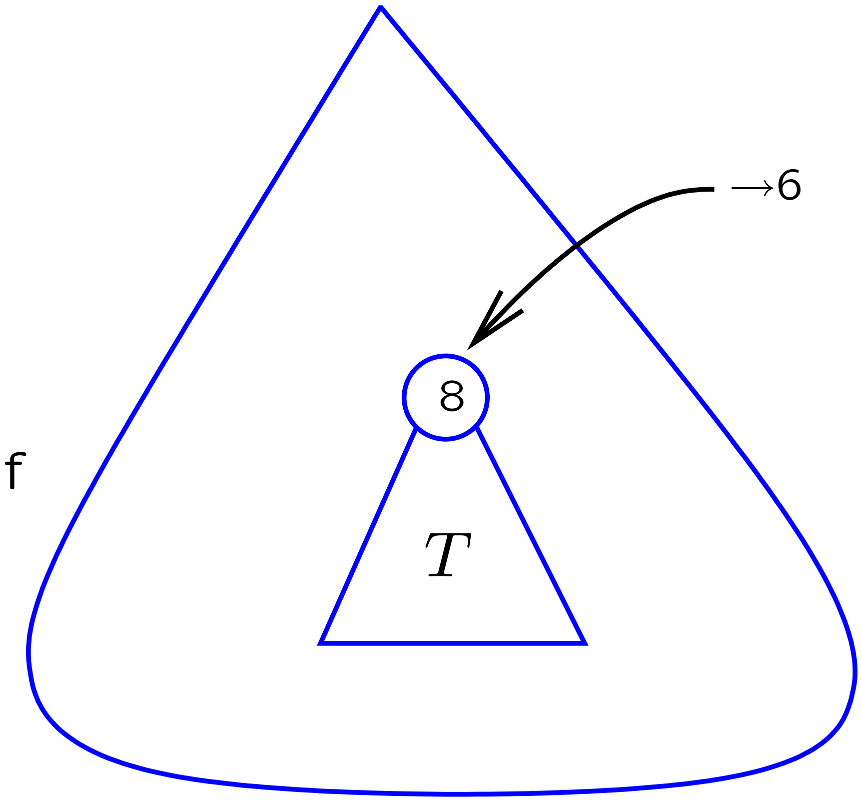
carries/borrows $\Rightarrow$ Use redundant zeroless number representation

$\cancel{1000_{\text{two}}}$ $\Rightarrow$ $32_{\text{redundant-four}}$

[ISAAC 2006]

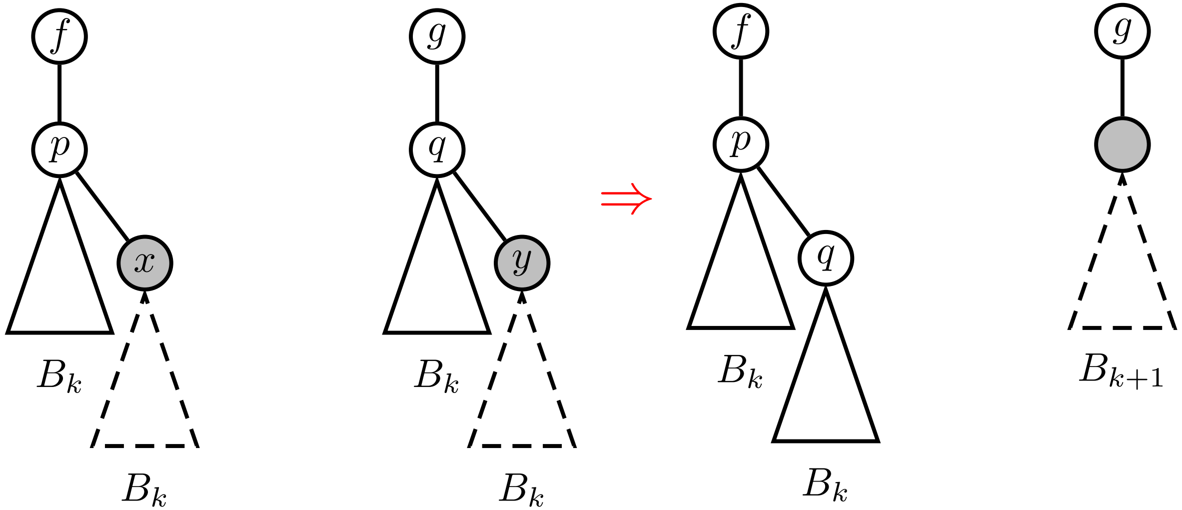
Decrease

- cut the subtree T
- put a phantom node instead
- make the element replacement
- see T as a separate tree
- reduce the # of phantom nodes if necessary



Data-structural transformations

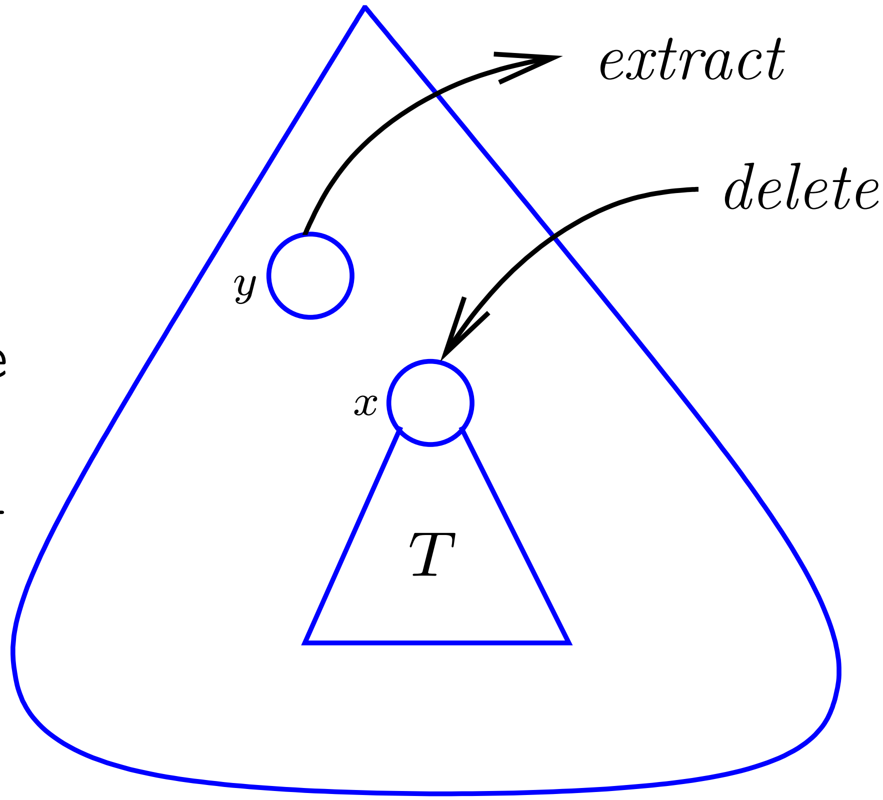
Singleton transformation I: Both x and y are the last children of their parents p and q, respectively. Name the nodes such that $\mathit{element}[p] \not> \mathit{element}[q]$.



\+ 4 other transformations, see the proceedings

Delete

- cut the subtree T rooted at x
- replace with a phantom node
- remove x
- extract a node y
- join the subtrees of T and y
- make the new tree as a separate tree
- update the minimum pointer if necessary
- reduce the # of phantom nodes if necessary



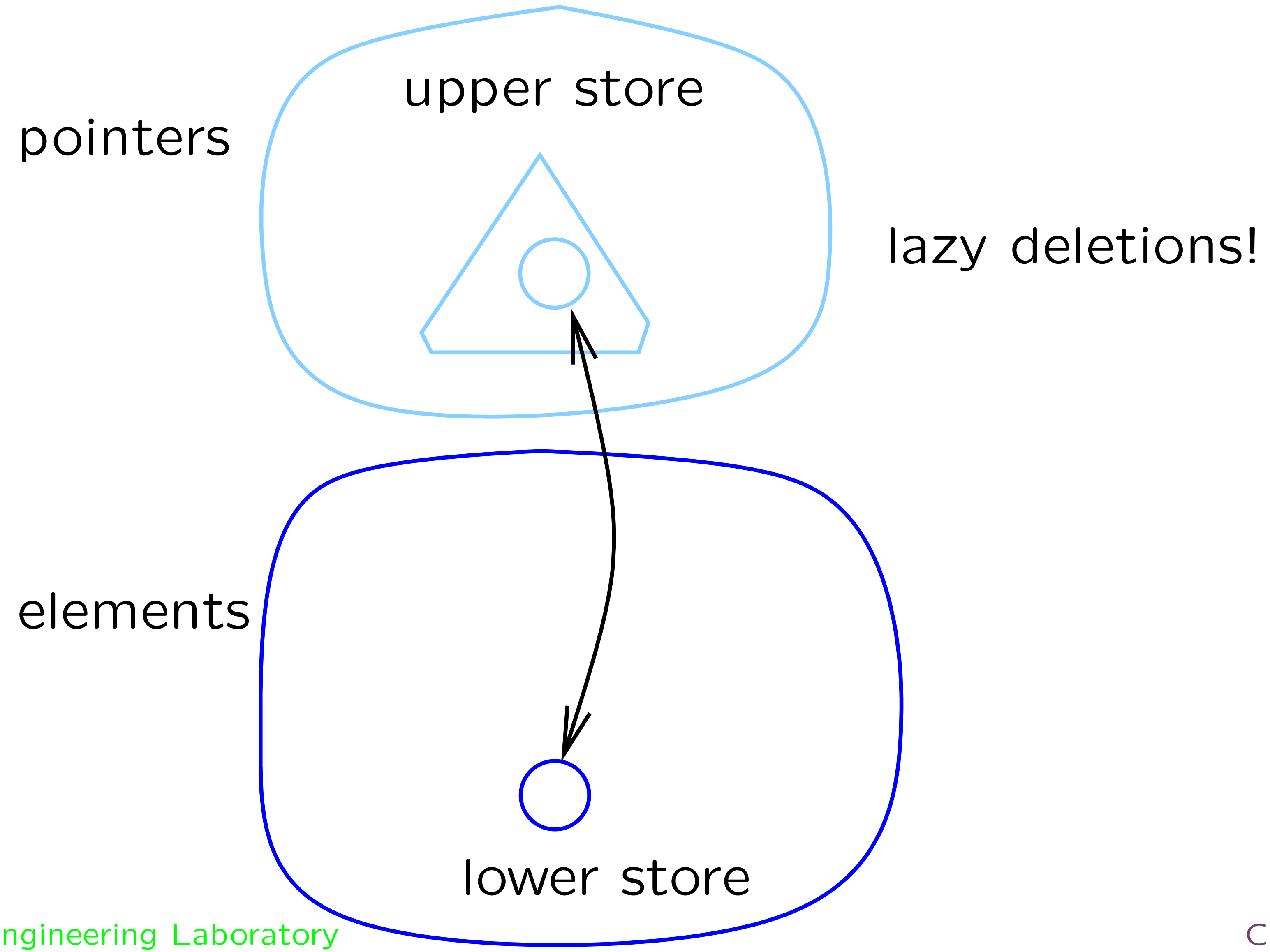
Analysis

Theorem: A node can have at most $\lg n + O(\sqrt{\lg n})$ real children

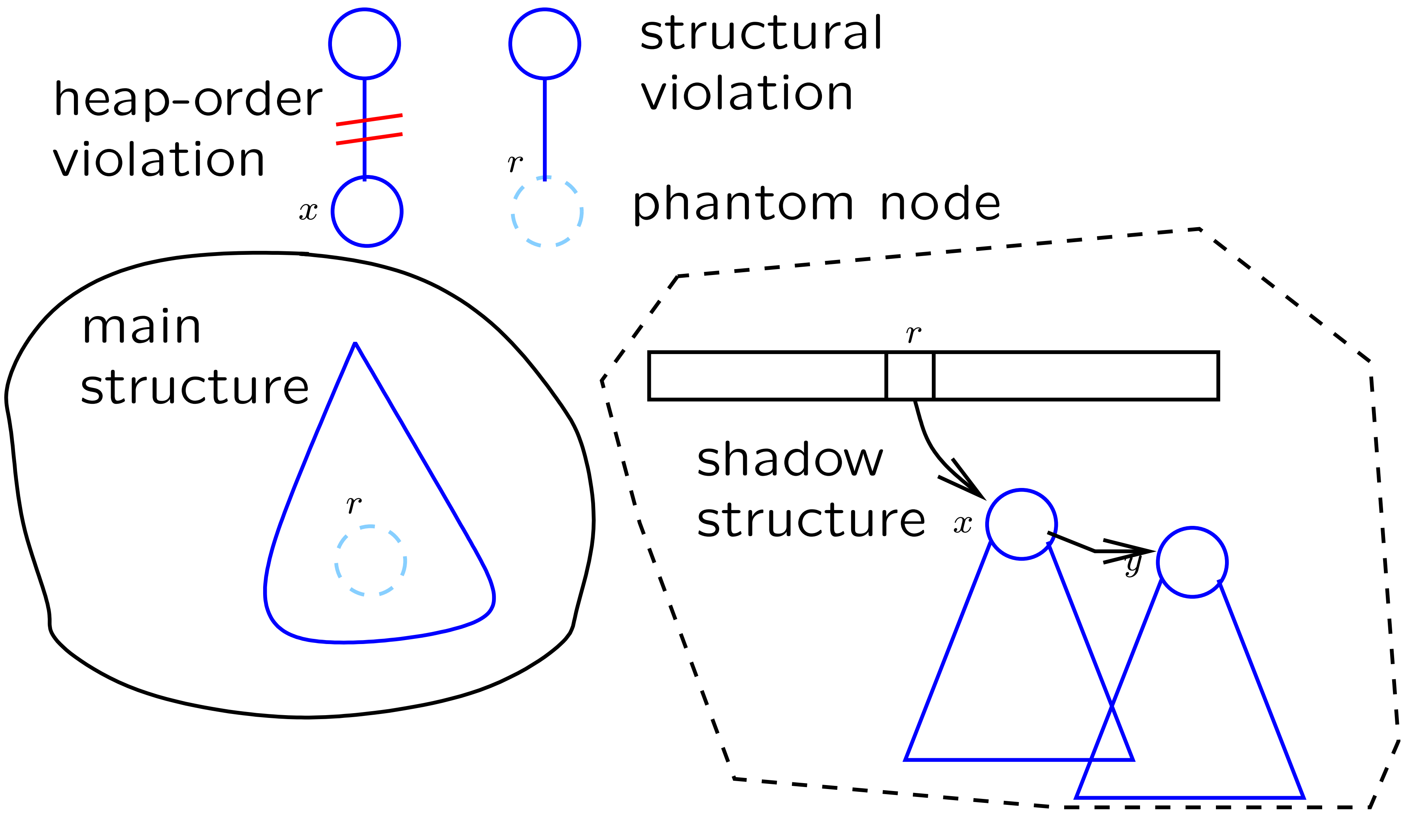
For a proof, see the proceedings

$\Rightarrow$ *delete* performs at most $2 \lg n + O(\sqrt{\lg n})$ element comparisons

Two-tier heaps



Mimicking heap-order violations



Main contribution

Theorem: Relaxed heaps (heap-order violations) and pruned heaps (structural violations) are equal in power up to $\lg n + O(\lg \lg n)$ element comparisons per $delete$

Open problems

- What is the answer to our original research question, i.e. is $\lg n + O(1)$ element comparisons per $delete()$ possible or not?
- Are the two types of violations in 1-1 correspondence or not?
- What is the lowest number of element comparisons performed by $delete$ for heaps that are efficiently meldable?
- How to implement a worst-case efficient heap in an industry-strength program library?