In-Place Data Structures: Which Complexity Measures Do Matter?

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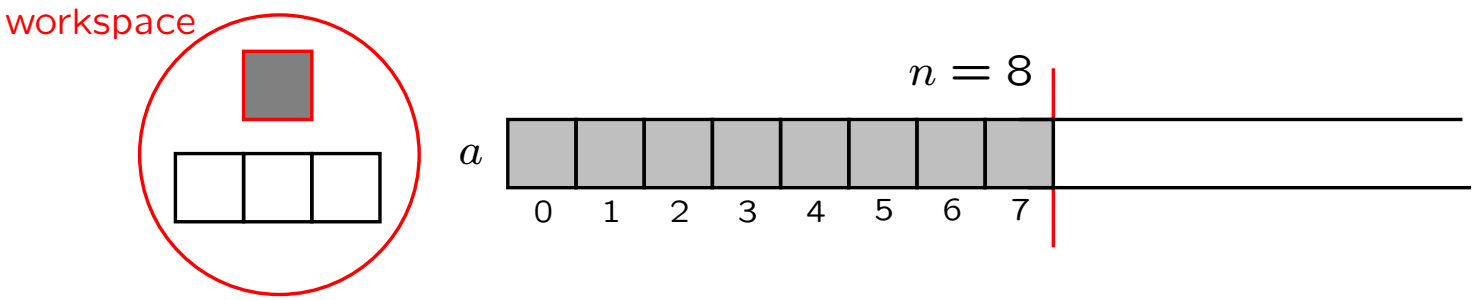
Model of computation

Available

- An **infinite** array a suitable for storing elements
- $O(1)$ number of other memory locations for storing elements
- $O(1)$ number of other variables (counters, indices, bit strings of length $\lceil \lg(1+n) \rceil$)



Requirement

- If the data structure stores n elements, these elements must be kept in the first n locations of a.

Coverage

In-place data structures

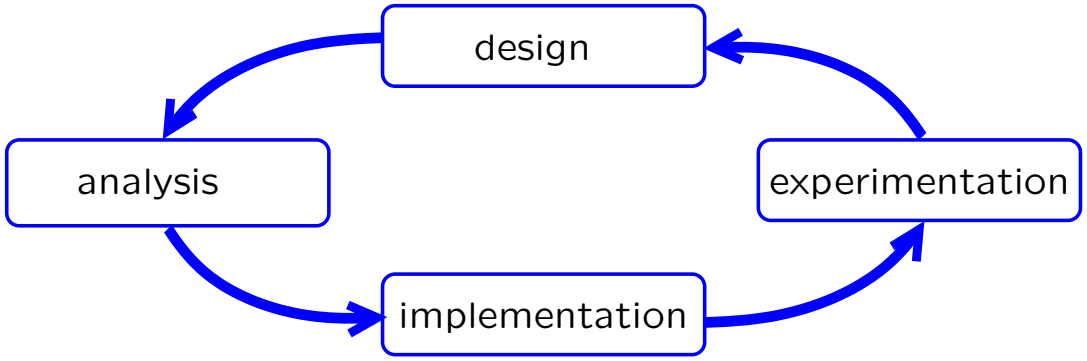
- Binary heaps
- Static search trees

Complexity measures

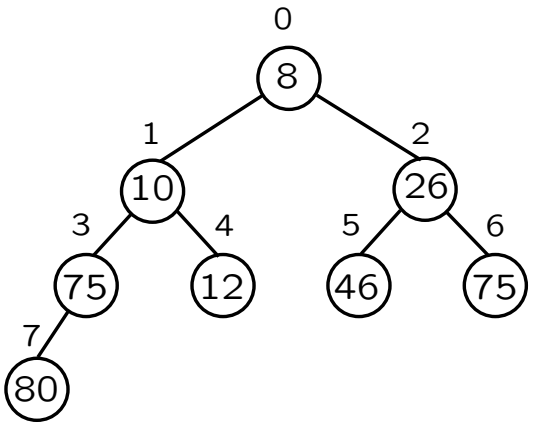
- Space utilization
- # Element comparisons
- # Element moves
- # Cache misses
- # Branch mispredictions
- Running time

Aha! The whole cycle

What is important?



Binary heaps



$n = 8$

a	8	10	26	75	12	46	75	80
	0	1	2	3	4	5	6	7

left-child(i)
    return $2i + 1$

right-child(i)
    return $2i + 2$

parent(i)
    return $\lfloor (i - 1)/2 \rfloor$

construct()
    for $(i = parent(n - 1);\ i \geq 0;\ --i)$
        siftdown(i)

minimum()
    return $a[0]$

insert(x)
    $a[n] = x$
    siftup(n)
    n += 1

extract-min()
    $min = a[0]$
    n −= 1
    $a[0] = a[n]$
    siftdown(0)
    return min

Experimental setup

Standard benchmark

– construct a heap of size n

Input data

All elements are of type `int`

Repetitions

Repeat each experiment r times, $r = 2^{26}/n$

Reported value

Measurement result divided by $r \times n$

Processor

Intel® Core™ i5-2520M CPU @ 2.50GHz × 4

Memory system

12-way-associative L3 cache: 3 MB

cache lines: 64 B

main memory: 3.8 GB

Operating system

Ubuntu 12.04 (Linux kernel 3.2.0-29-generic)

Compiler

`g++` compiler (`gcc` version 4.6.3) with optimization `-O3`

Reduce # element comparisons

Inventor	*construct*	*insert*	*extract-min*	Extra Space
Williams/Floyd	$2n$	$\sim \lg n$	$\sim 2 \lg n$	$O(1)$ words
Gonnet & Munro	$1.625n$			$\Theta(n)$ words
Gonnet & Munro		$\sim \lg \lg n$	$\sim \lg n + \log^* n$	$O(1)$ words
Lower bounds	$\sim 1.37n$	$\Omega(1)$	$\sim \lg n$	$\Omega(1)$ words

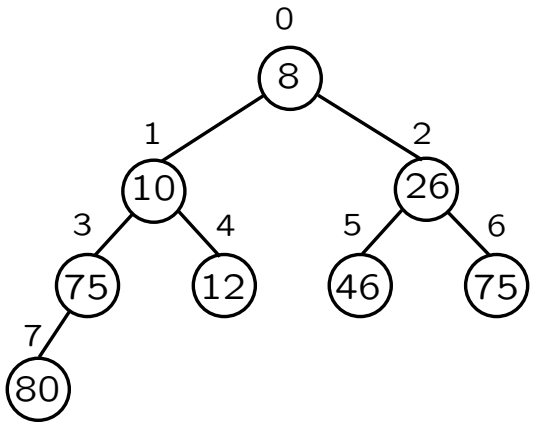
construct: Use a binomial tree in the construction

insert: Binary search on the *siftup* path

extract-min: $\lg n - \lg \lg n$ levels down along the *siftdown* path, *siftup* or recur further down

Floyd's heap-construction program

```
 1 template <typename position, typename index, typename comparator>
 2 void siftdown(position a, index i, index n, comparator less) {
 3   typedef typename std::iterator_traits<position>::value_type element;
 4   element copy = a[i];
 5 loop:
 6   index j = 2 * i;
 7   if (j <= n) {
 8     if (j < n)
 9       if (less(a[j], a[j + 1]))
10         j = j + 1;
11     if (less(copy, a[j])) {
12       a[i] = a[j];
13       i = j;
14       goto loop;
15     }
16   }
17   a[i] = copy;
18 }
19
20 template <typename position, typename comparator>
21 void make_heap(position first, position beyond, comparator less) {
22   typedef typename std::iterator_traits<position>::difference_type index;
23   position const a = first - 1;
24   index const n = beyond - first;
25   for (index i = n / 2; i > 0; --i)
26     siftdown(a, i, n, less);
27 }
```



$n = 8$

a	8	10	26	75	12	46	75	80
	0	1	2	3	4	5	6	7

[Floyd 1964]

Remove an easy-to-predict `if`

opt$_1$: Make sure that `siftdown` is always called with an odd `n`

```
if (j < n)
  ...
for (index i = n / 2; i > 0; --i)
  siftdown(a, i, n, less);
```

$\longrightarrow$

```
template <typename position, typename index, typename comparator>
void siftup(position a, index j, comparator less) {
  ...
}

index const m = (n & 1) ? n : n - 1;
for (index i = m / 2; i > 0; --i)
  siftdown(a, i, m, less);
siftup(a, n, less);
```

Construction time [ns]

n	F	F_1
2^{10}	7.5	7.1
2^{15}	7.4	7.0
2^{20}	8.2	7.9
2^{25}	8.9	8.4

Remove a hard-to-predict `if`

opt$_2$: Interpret the result of a comparison as an integer and use this value in normal index arithmetic

```
if (condition) {
  j = j + 1;
}
```

$\longrightarrow$

```
j = j + condition;
```

Construction time [ns]

n	$\mathbf{F}_1$	$\mathbf{F}_{12}$
2^{10}	7.1	4.8
2^{15}	7.0	4.9
2^{20}	7.9	6.3
2^{25}	8.4	7.2

Lean programs

- A program has a **constant** number of unnested loops.
- Each loop is **branch-free**, except the final conditional branch at the end.
- A branch predictor is **static**: forward branches are not taken and backward branches are taken.
- Each such program induces $O(1)$ branch mispredictions in this model.

Theorem. Let $\mathcal{P}$ be a program of length κ, measured in the number of assembly-language instructions. Assume that the running time of $\mathcal{P}$ is $t(n)$ for an input of size n. There exists a program $\mathcal{Q}$ of length $O(\kappa)$ that is equivalent to $\mathcal{P}$, runs in $O(\kappa t(n))$ time for the same input as $\mathcal{P}$, and induces $O(1)$ branch mispredictions.

[Elmasry, Katajainen 2012]

Reduce # element moves

opt$_3$: Do not make any element moves when the element at the root stays in its original location

```
element copy = a[i];
```

$\longrightarrow$

```
element copy;
index k = 2 * i;
k = k + less(a[k], a[k + 1]);
if (less(a[i], a[k])) {
  copy = a[i];
  a[i] = a[k];
}
else {
  return;
}
i = k;
```

Aha! Loop unrolling

Construction time [ns]

n	F$_{12}$	F$_{123}$
2^{10}	4.8	4.3
2^{15}	4.9	4.6
2^{20}	6.3	5.9
2^{25}	7.2	6.9

Element moves

n	F	F$_{123}$
2^{10}	1.73	1.52
2^{15}	1.74	1.53
2^{20}	1.74	1.53
2^{25}	1.74	1.52

Reduce # cache misses

opt$_4$: Visit the nodes in reverse depth-first order instead of reverse breadth-first order [Bojesen et al. 2000]

```
for (index i = n / 2; i > 0; --i)
  siftdown(a, i, n, less);
```

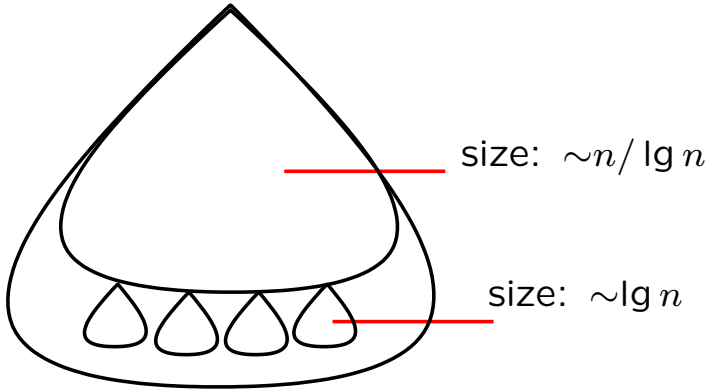
$\longrightarrow$

```
index j = n / 2;
index const i = j / 2;
while (j > i) {
  siftdown(a, j, n, less);
  index z = j;
  while ((z & 1) == 0) {
    z /= 2;
    siftdown(a, z, n, less);
  }
  --j;
}
```

Construction time [ns]

n	F	F$_{123}$	F$_{1\text{-}4}$
2^{10}	7.4	4.3	5.2
2^{15}	7.4	4.6	5.1
2^{20}	8.2	5.9	5.2
2^{25}	8.7	6.9	5.1

Making the GM algorithm in-place



1. Improve **GM**:
   $O(n)$ words $\longrightarrow$ $O(n)$ bits
2. Apply the improved algorithm for all bottom trees; keep the bits needed compactly in a word
3. Use **F**'s *siftdown* approach for the top tree.

Element comparisons

$\sim 2n \longrightarrow \sim 1.625n$

Element moves

$\sim 2n \longrightarrow \sim 2.125n$

Cache misses

$\sim \frac{n \lg B}{B} \longrightarrow \sim \frac{n}{B}$, assuming that $B \lg n << M$ (B block size; M memory size)

Construction time [ns]

n	**F**	**GM**
2^{10}	7.4	8.0
2^{15}	7.4	7.7
2^{20}	8.2	7.7
2^{25}	8.7	7.7

Heap construction: Summary

Construction time [ns]

n	std	F	F_{123}	$F_{1\text{-}4}$	GM
2^{10}	10.7	7.4	4.3	5.2	8.0
2^{15}	10.4	7.4	4.6	5.1	7.7
2^{20}	11.0	8.2	5.9	5.2	7.7
2^{25}	11.5	8.7	6.9	5.1	7.7

Instructions

n	std	F	F_{123}	$F_{1\text{-}4}$	GM
2^{15}					
2^{20}	35.5	20.8	13.4	16.2	42.9
2^{25}					

Element comparisons

n	std/F	GM
2^{10}	1.98	1.80
2^{15}	1.99	1.66
2^{20}	1.99	1.63
2^{25}	2	1.63

Branches | mispredictions

n	std	F	F_{123}	$F_{1\text{-}4}$	GM
2^{10}	5.39 | 0.96	4.53 | 0.81	2.17 | 0.27	2.42 | 0.47	3.60 | 0.66
2^{15}	5.40 | 0.89	2.43 | 0.78	2.18 | 0.24	2.43 | 0.47	2.39 | 0.38
2^{20}	5.41 | 0.89	4.57 | 0.78	2.18 | 0.24	2.43 | 0.47	– | –
2^{25}	5.41 | 0.89	4.56 | 0.78	2.18 | 0.24	2.43 | 0.47	– | –

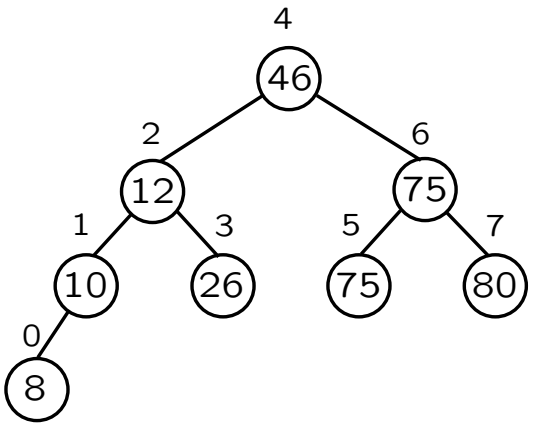
Element moves

n	std	F	GM
2^{10}	3.99	1.99	2.15
2^{15}	3.99	1.99	2.39
2^{20}	4	1.99	2.38
2^{25}	4	2	2.38

I/Os | misses (per n/B)

n	std/F	$F_{1\text{-}4}$	GM
2^{10}	1.00 | 1.00	1.00 | 1.00	0.95 | 0.95
2^{15}	5.66 | 1.00	1.03 | 1.00	1.03 | 1.00
2^{20}	5.87 | 4.94	1.04 | 1.00	– | –
2^{25}	5.87 | 5.84	1.04 | 0.99	– | –

Static search trees



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a	8	10	12	26	46	75	75	80
	0	1	2	3	4	5	6	7

$left\text{-}child(i)$
    return ...

$right\text{-}child(i)$
    return ...

$construct()$
    $sort(a, a + n)$

$is\text{-}member(x)$
    $i = 0$
    $k = n$
    while $i \neq k$
        if $x < a[i]$
            $k = i$
            $i = left\text{-}child(i)$
        else if $a[i] < x$
            $i = right\text{-}child(i)$
        else
            return yes
    return no

Static search trees vs. red-black trees

Standard benchmark

r random $is\text{-}member$ queries, $r = 10^6$

Input data

All elements are of type `int`

Reported value

Measurement result divided by $r \times \lg n$

Search time [ns]

n	Sorted array	Red-black tree
2^{10}	6.8	5.5
2^{15}	6.9	10.9
2^{20}	14.7	36.6
2^{25}	32.1	64.3

Idea: Rely on two-way element comparisons

Three-way comparisons

```
is-member(x)
    i = 0
    k = n
    while i ≠ k
        if x < a[i]
            k = i
            i = left-child(i)
        else if a[i] < x
            i = right-child(i)
        else
            return yes
    return no
```

Two-way comparisons

```
is-member(x)
    i = 0
    j = −1
    k = n
    while i ≠ k
        if x < a[i]
            k = i
            i = left-child(i)
        else
            j = i
            i = right-child(i)
    if j == −1 or a[j] < x
        return no
    return yes
```

[Andersson 1991]

Performance of the optimized program

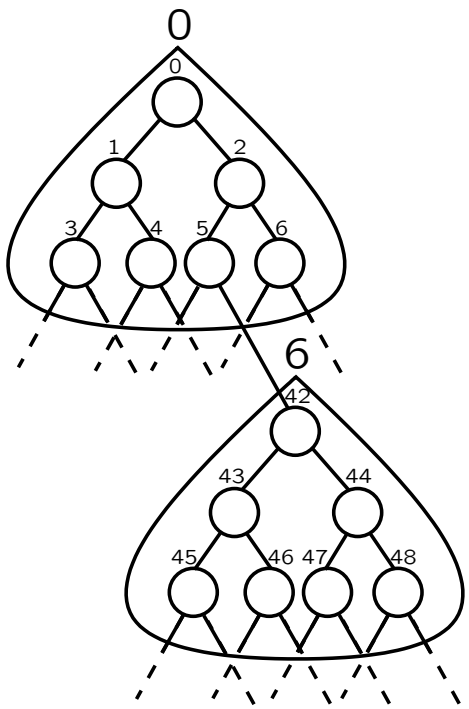
Element comparisons

$$\sim 2 \lg n \longrightarrow \sim \lg n$$

Search time [ns]

n	Three-way	Two-way
2^{10}	6.6	5.5
2^{15}	7.7	6.9
2^{20}	15.6	14.7
2^{25}	33.3	32.1

Idea: Use another memory layout



$F = 7$

left-child(i)
    $j = i/F$
    if $i < \lfloor F/2 \rfloor + j * F$
        return $2 * i - j * F + 1$
    else
        return $F * (2 * i - (1 - F) * j - F + 2)$

right-child(i)
    // Left as an exercise

construct$()$
    // More complicated than sorting
    ...

is-member(x)
    // As before

Performance of implicit local search trees

Fat-node visits

$\sim \lg n - \lg F \longrightarrow \sim \lg n / \lg F$,
where F is the size of the fat nodes measured in elements

Search time [ns]; $F = 15$

n	Sorted array	Implicit local
2^{10}	5.5	15.0
2^{15}	6.9	15.0
2^{20}	14.7	16.3
2^{25}	32.1	20.1

Cache behaviour

All values are divided by $r \times \log_B n$, where B is the number of elements that fit in a cache line (16 in our test)

n	Sorted array			Implicit local		
	Refs.	I/Os	Misses	Refs.	I/Os	Misses
2^{10}	7.40	0.00	0.00	10.56	0.00	0.00
2^{15}	6.93	2.00	0.00	9.46	0.39	0.00
2^{20}	6.70	3.20	0.73	8.80	0.81	0.12
2^{25}	6.56	3.37	3.01	9.00	1.01	0.51

Idea: Avoid conditional branches

Two-way comparisons

```
is-member(x)
    j = −1
    i = 0
    while i < n
        if x < a[i]
            i = left-child(i)
        else
            j = i
            i = right-child(i)
    if j == −1 or a[j] < x
        return no
    return yes
```

Hard-to-predict `if` removed

```
choose(condition, i, j)
    return j + condition * (i − j)

is-member(x)
    j = −1
    i = 0
    while i < n
        smaller = x < a[i]
        j = choose(smaller, j, i)
        i = choose(smaller, left-child(i), right-child(i))
    if j == −1 or a[j] < x
        return no
    return yes
```

Performance of the `if`-free programs

Conditional branches

$$\sim 2 \lg n \longrightarrow \sim \lg n$$

Branch mispredictions

$$\sim 0.5 \lg n \longrightarrow O(1)$$

Search time [ns]; $F = 15$

n	Sorted array		Implicit local	
	Two-way	`if`-free	Two-way	`if`-free
2^{10}	5.5	6.6	15.0	15.0
2^{15}	6.9	7.2	15.0	14.2
2^{20}	14.7	20.7	16.3	15.6
2^{25}	32.1	45.8	20.1	22.8

Branch behaviour

n	Sorted array Two-way		Sorted array `if`-free		Implicit local Two-way		Implicit local `if`-free	
	⩽	Mispred.	⩽	Mispred.	⩽	Mispred.	⩽	Mispred.
2^{10}	2.20	0.62	1.20	0.10	3.56	0.89	1.32	0.11
2^{15}	2.07	0.57	1.13	0.07	3.21	0.86	1.12	0.07
2^{20}	2.05	0.55	1.10	0.05	3.05	0.84	1.05	0.05
2^{25}	2.04	0.54	1.08	0.04	3.18	0.82	1.09	0.04

Conclusions

- Branch optimization is only effective for small problem instances
- There is no reason to remove easy-to-predict conditional branches
- Cache optimization is effective for large problem instances, but it may make the solution slower for small problem instances
- It would be cool if branch optimization was done automatically by the compiler
- Element comparisons and element moves are still relevant in the cases where they are expensive

What else?

and open questions

- Devise an in-place priority queue for which $insert$ requires $O(1)$ worst-case time and $extract\text{-}min$ $O(\lg n)$ worst-case time including at most $\lg n + O(1)$ element comparisons.
- Can you improve the bounds for heap construction?

Some relevant papers

Binary heaps

- Jesper, Jyrki, Maz: Performance engineering case study: Heap construction, WAE 1999
- Claus, Jyrki, Fabio: Experimental evaluation of local heaps, CPH STL Report 2006-1
- Jingsen, Stefan, Amr, Jyrki: In-place heap construction with optimized comparisons, moves, and cache misses, MFCS 2012
- Amr, Jyrki: Bypassing the lower bounds for binary heaps

Branch prediction

- Amr, Jyrki: Lean programs, branch mispredictions, and sorting, FUN 2012
- Amr, Jyrki, Max: Branch mispredictions don't affect mergesort, SEA 2012
- Amr, Jyrki: Microbenchmarking the search procedure for balanced search trees