Weak Heaps and Friends: Recent Developments

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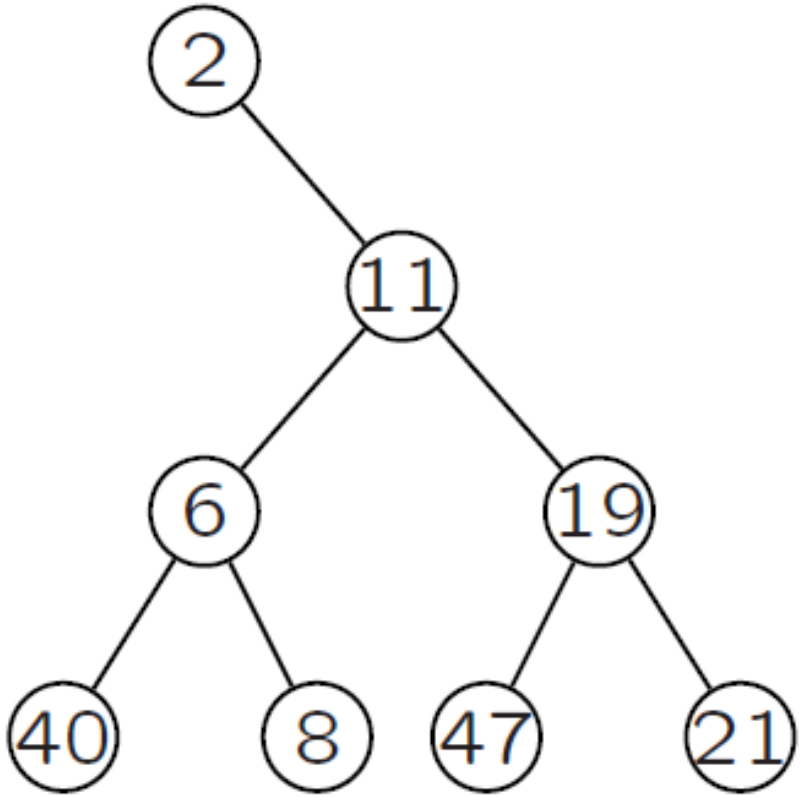
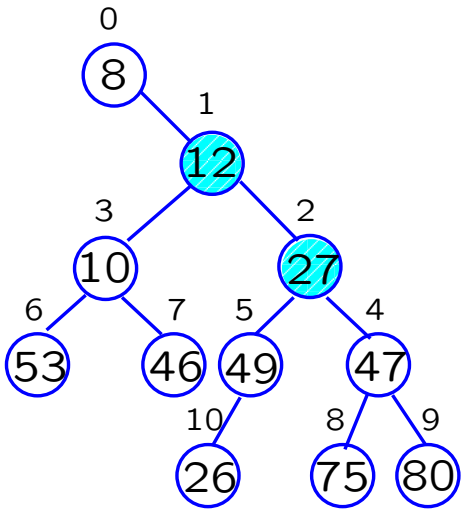
These are Stefan's slides for his invited talk at IWOCA 2013

(Complete) Weak Heap



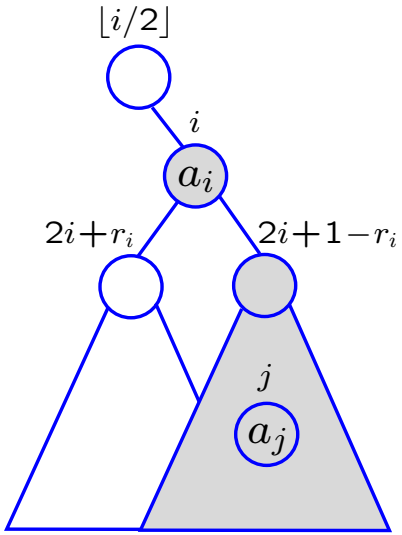
Array-Based Representation



Array a of elements

Array r of bits (1's in cyan)

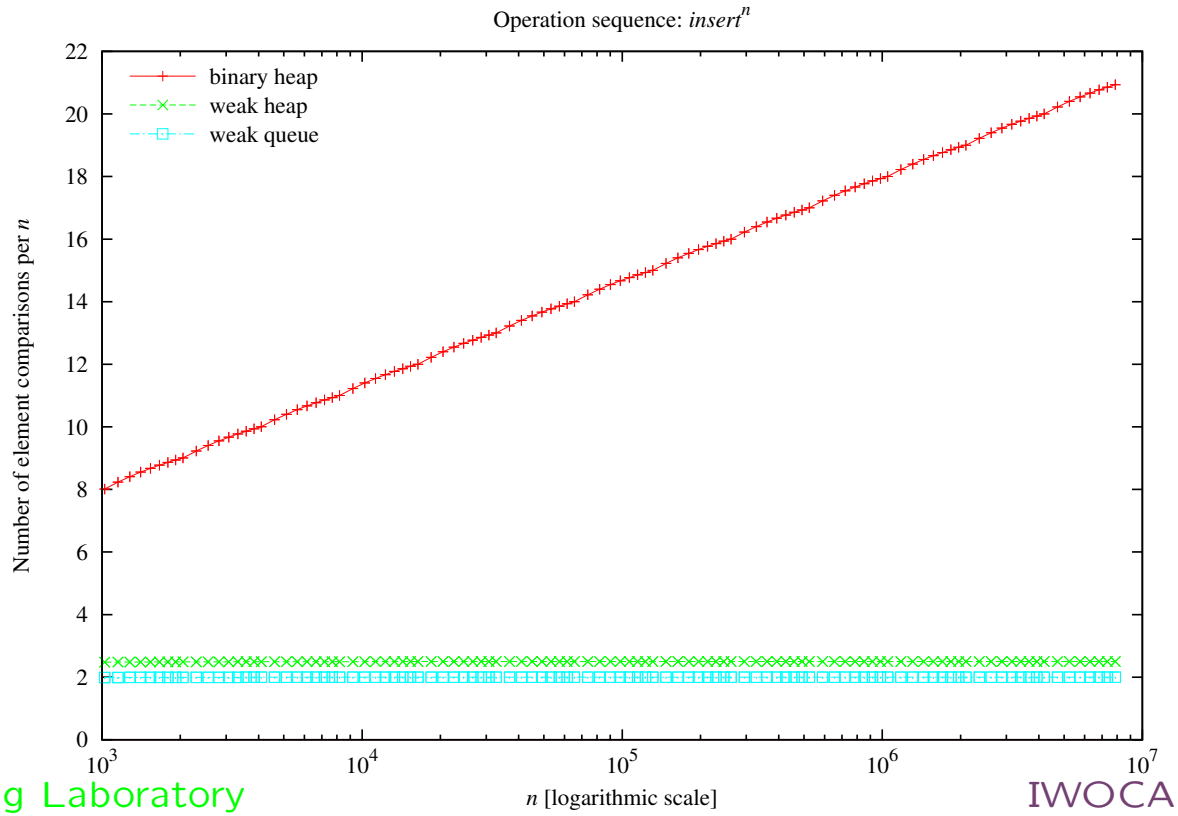
0	1	2	3	4	5	6	7	8	9	10
8	12	27	10	47	49	53	46	75	80	26



Why Weak Heaps?

Data structure	*construct*	*minimum*	*insert*	*extract-min*
binary heap [Flo64,Wil64]	$2n$	0	$\lceil \lg n \rceil$	$2\lceil \lg n \rceil$
weak heap [Dut93]	$n-1$	0	$\lceil \lg n \rceil$	$\lceil \lg n \rceil$

Repeated insertions [IWOCA-12, JDA-13]



Structure of the Talk: Research Questions

What is the "best" heap-construction algorithm?

What is the "best" sorting algorithm?

What is the "best" priority queue?

What is the best in-place heap-construction algorithm?

Best $\sim$ In terms of element comparisons and practical running time

In-place $\sim \Theta(1)$ extra words

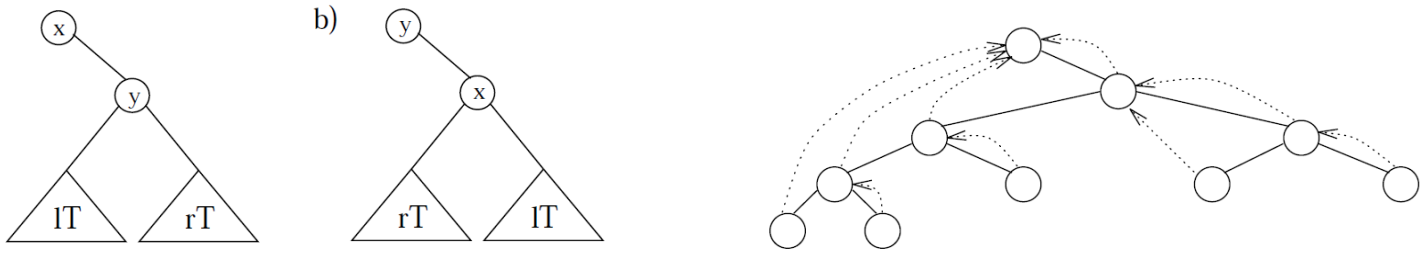
Some Options

Element comparisons

Inventor	Abbreviation	Worst	Average	Extra space
Floyd	**alg. F**	$2n$	$\sim 1.88n$	$\Theta(1)$ words
Gonnet & Munro	**alg. GM**	$\sim 1.625n$	$\sim 1.625n$	$\Theta(n)$ words
McDiarmid & Reed	**alg. MR**	$2n$	$\sim 1.52n$	$\Theta(n)$ bits
Li & Reed	**lower bound**	$\sim 1.37n$	$\sim 1.37n$	$\Omega(1)$ words

Average-case results assume that the input is a random permutation of n distinct elements

Building Binary Heaps



Weak heap: Lower bound $n - 1$ (element comparisons)

Weak heap $\rightarrow$ binary heap: $\sim 0.625n$ [IWOCA-12, MFCS-12]

$\rightsquigarrow$ $1.625n$ heap construction, n bits (worst case)

Bottom trees: $\rightsquigarrow$ $1.625n$ in-place heap construction (worst case)

[$\rightsquigarrow$ $1.52n$ in-place heap construction (average case)]

Weak Heap -> Binary Heap

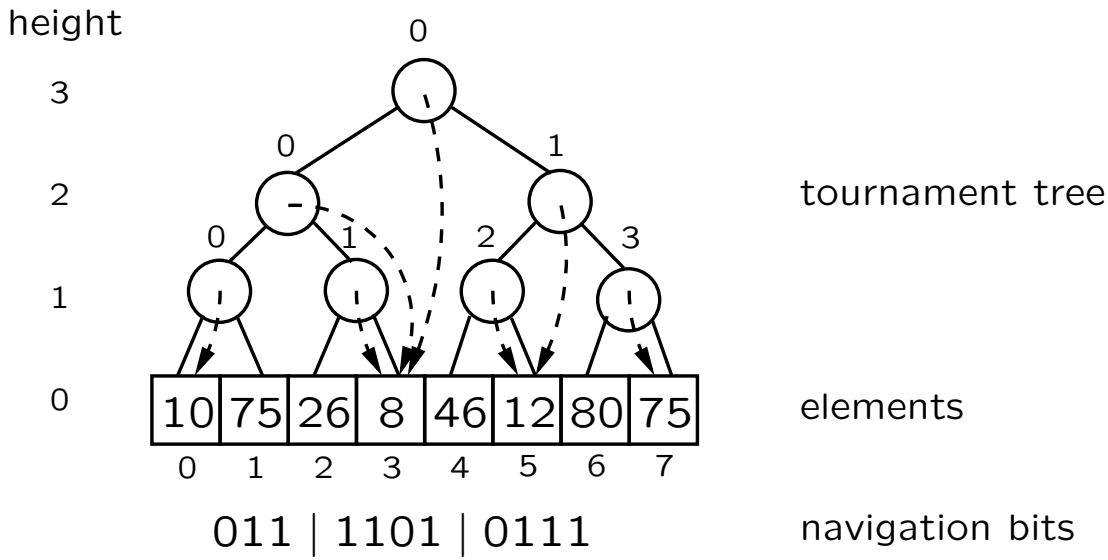
GM: Build a binary heap in two phases: 1) Construct a heap-ordered binomial tree 2) Convert this tree into a binary heap

Alternative: a **complete weak heap** $\rightarrow$ # element comparisons

$$\begin{aligned} C(8) &= 1, \\ C(2^k) &= 2C(2^{k-1}) + k - 1 \end{aligned}$$

For $n = 2^k \geq 8$, the solution of this relation is $C(n) = 5/8 \cdot n - \lg n - 1$

Alternative: a **navigation pile** $\rightarrow$ less element moves



Bottom-Tree Conversion

Bottom trees: All complete binary trees of size $m = 2^{\lfloor \lg \lg n \rfloor + 1} - 1$

1. Convert all bottom trees to **bottom heaps**
2. Ensure heap order at upper levels by using Floyd's sift-down procedure
3. Optimize element moves by handling binary **micro trees** of size 7 differently

- Elements involved in all bottom-heap constructions $\leq n$
  $\rightarrow$ $1.625n$ element comparisons
- At most $\lceil n/2^{h+1} \rceil$ nodes of height h
  $\rightarrow$ $o(n)$ element comparisons at the levels above the bottom trees

Experimental Setup and Summary

Random permutations of n distinct int(eger)s for different (small, medium, large, and very large) problem sizes

Programs tuned to construct binary heaps of size $2^k - 1$

- **GM** showed acceptable practical performance
- number of element comparisons and element moves was larger for **in-situ GM** than for **in-situ MR**
- **in-situ GM** was faster than **in-situ MR**
- but beaten by **F** and its **BKS** variant

Element Comparisons

n	std	F	BKS	in-situ GM	in-situ MR
$2^{10}-1$	1.64	1.86	1.86	1.74	1.52
$2^{15}-1$	1.64	1.88	1.88	1.65	1.54
$2^{20}-1$	1.64	1.88	1.88	1.63	1.53
$2^{25}-1$	1.65	1.88	1.88	1.63	1.53

std: Bottom-up heap construction (`make_heap`, Floyd, Wegener)

BKS: Improved version of Floyd's algorithm (Bojesen et al. [JEA-00])

Execution Times

n	std	F	BKS	in-situ GM	in-situ MR
$2^{10}-1$	22.3	14.6	17.1	21.3	26.2
$2^{15}-1$	22.2	14.6	17.4	23.0	24.4
$2^{20}-1$	29.3	21.9	17.8	22.9	23.6
$2^{25}-1$	29.8	21.7	17.5	22.9	23.6

std: Bottom-up heap construction (`make_heap`, Floyd, Wegener)

BKS: Improved version of Floyd's algorithm (Bojesen et al. [JEA-00])

What is the best constant-factor-optimal in-situ/adaptive sorting algorithm?

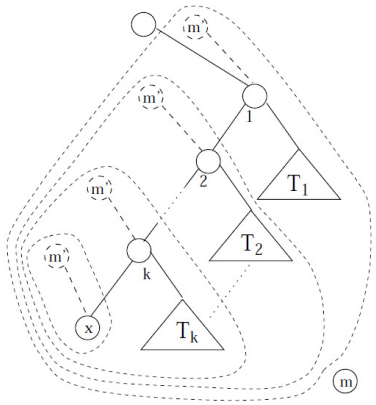
Best $\sim$ In terms of element comparisons and practical running time

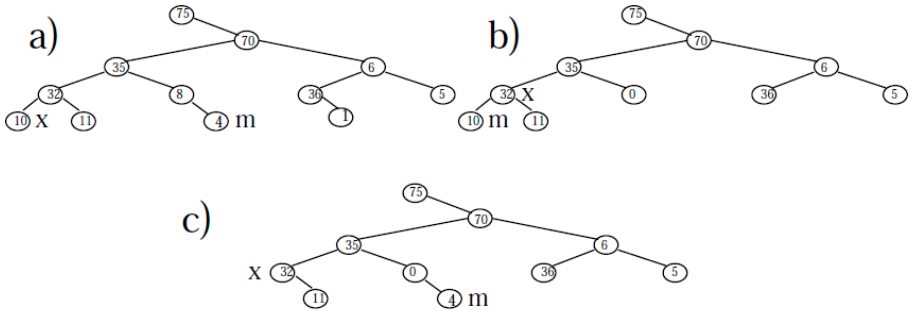
In-situ $\sim \Theta(\lg n)$ extra words

Adaptive $\sim$ with respect to inversions

Sequential Sorting

Lower bound: $\lg n! = n \lg n - n/\ln 2 + O(\lg n)$, where $1/\ln 2 = 1.4426$





- **Worst case:** $n \lg n + 0.1n$ [Dutton 1993, BIT]
- **Best case/index sorting:** $n \lg n - 0.9n$ [STACS-00, JEA-02]
- **QuickWeakHeapsort:** $n \lg n + 0.2n$ on average, in-place [JEA-02]
- **Optimal adaptive sorting:** $n \lg(Inv(n)/n) + O(n)$ worst case, two options [IWOCA-11, JDA-12]

Constant-Factor-Optimal Algorithms

	Space	Time	Worst	Average	Observed
Lower bound	$O(1)$	$\Omega(n \lg n)$	-1.44	-1.44	
BUHeapsort [Weg93]	$O(1)$	$O(n \lg n)$	$\omega(1)$	–	[0.35,0.39]
WeakHeapsort [Dut93]	$O(n/w)$	$O(n \lg n)$	0.09	–	[-0.46,-0.42]
RWeakHeapsort [ES02]	$O(n)$	$O(n \lg n)$	-0.91	-0.91	-0.91
Mergesort [Knu73]	$O(n)$	$O(n \lg n)$	-0.91	-1.26	–
EWeakHeapsort	$O(n)$	$O(n \lg n)$	-0.91	**-1.26**	–
Insertionsort [Knu73]	$O(1)$	$O(n^2)$	-0.91	**-1.38**	–
MergeInsertion [Knu73]	$O(n)$	$O(n^2)$	-1.32	**-1.3999**	**[-1.43,-1.41]**
InPlaceMergesort [R92]	$O(1)$	$O(n \lg n)$	-1.32	–	–
QuickHeapsort [DW13]	$O(1)$	$O(n \lg n)$	$\omega(1)$	-0.03	$\approx$ 0.20
	$O(n/w)$	$O(n \lg n)$	$\omega(1)$	-0.99	$\approx$ -1.24
QuickMergesort (IS)	$O(\lg n)$	$O(n \lg n)$	-0.32	**-1.38**	–
QuickMergesort	$O(1)$	$O(n \lg n)$	-0.32	**-1.26**	**[-1.29,-1.27]**
QuickMergesort (MI)	$O(\lg n)$	$O(n \lg n)$	-0.32	**-1.3999**	**[-1.41,-1.40]**

Idea of QuickXsort

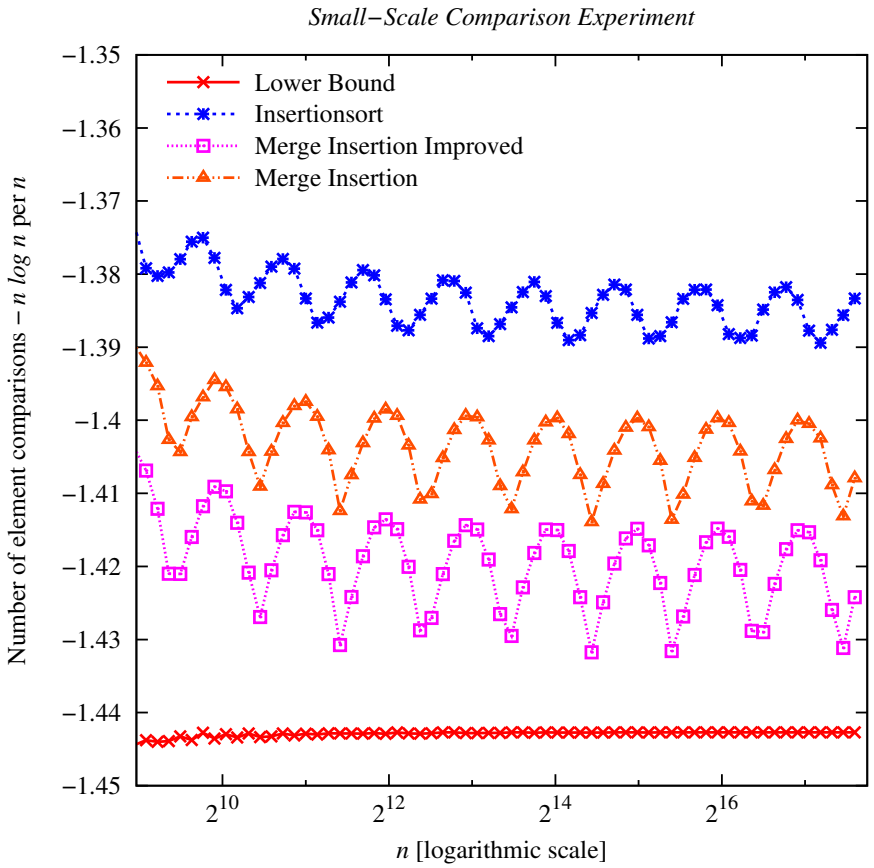
As in **Quicksort** the array is partitioned into the elements greater and less than some pivot element

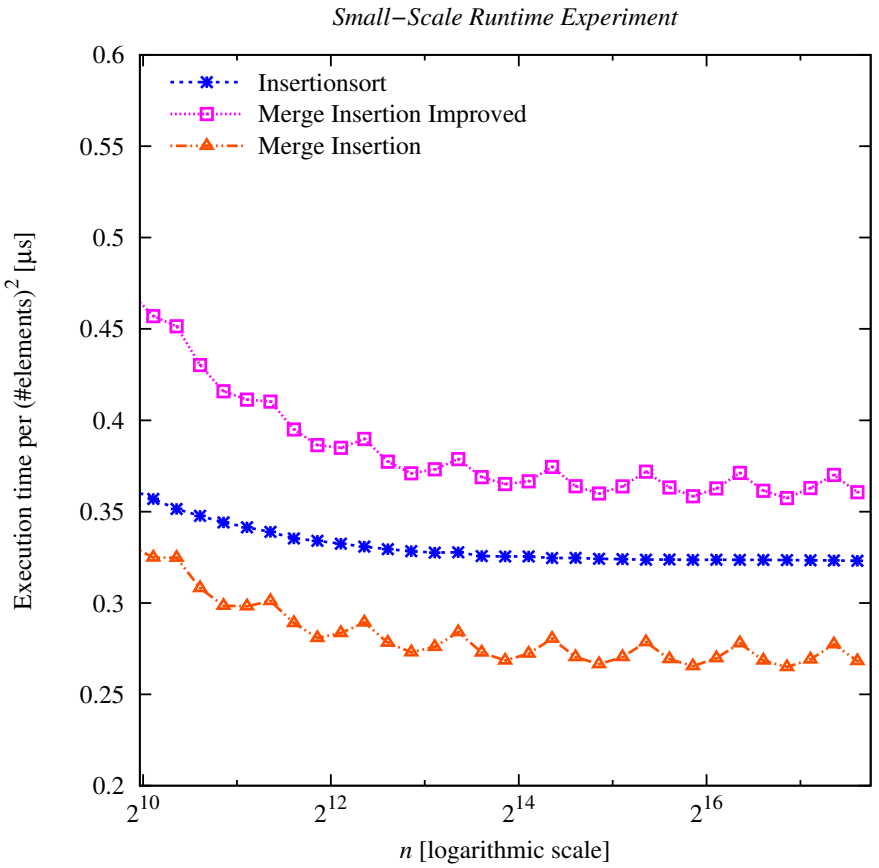
Then one part of the array is sorted by some algorithm **X** and the other part is sorted recursively

The advantage of this procedure is that, if **X** is a black box, then in **QuickXsort** the part of the array which is not currently being sorted may be used as temporary space, what yields an **in-situ** variant of **X**

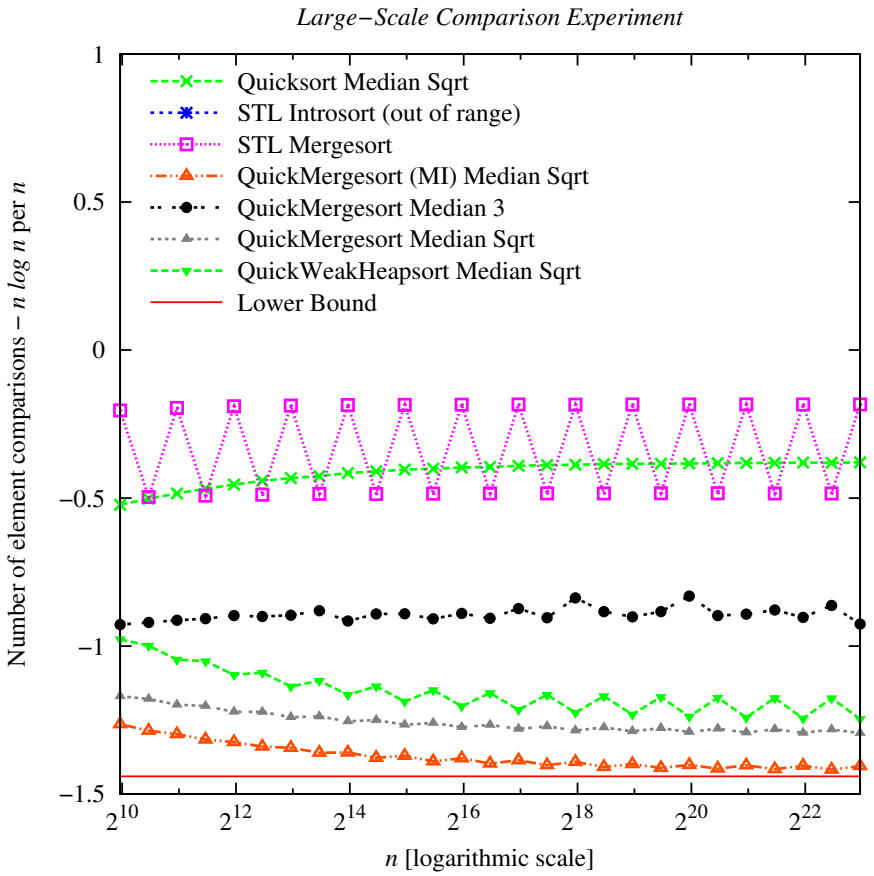
By taking a sample of $\Theta(\sqrt{n})$ elements when selecting the pivot, **QuickXsort** performs, on an average, the same number of element comparisons as **X** up to an $o(n)$ lower-order term

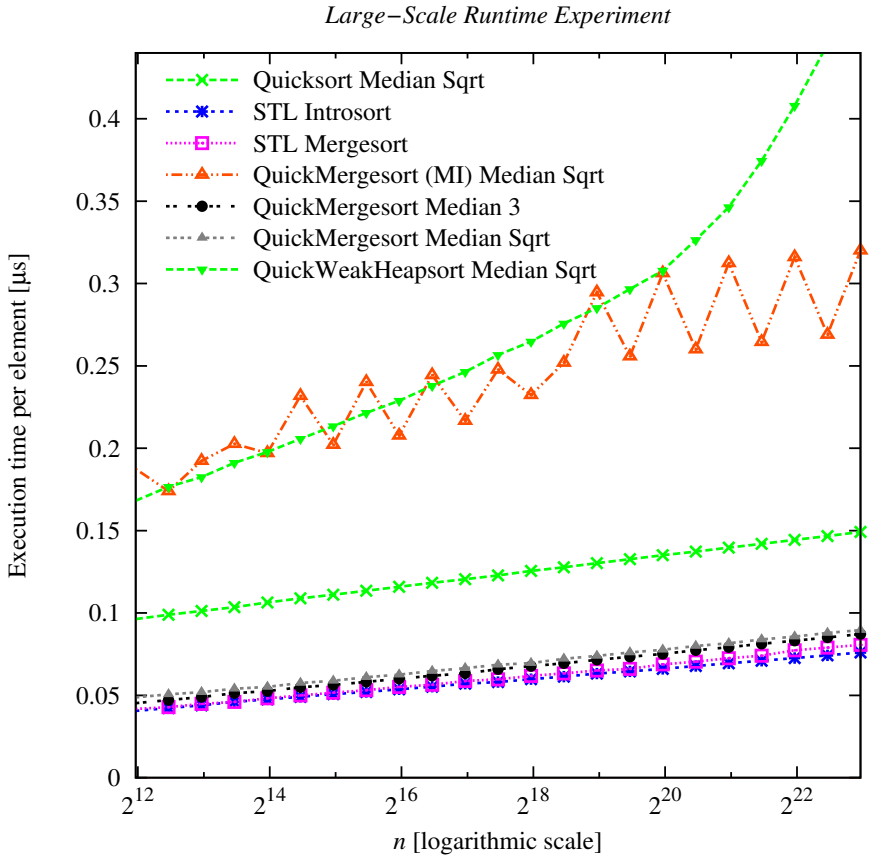
Results for Small Datasets





Results for Large Datasets





Adaptive Sorting

- A sorting algorithm is **adaptive** with respect to a measure of disorder, if it sorts all input sequences, but performs particularly well on those that have a low amount of disorder.
- The running time of such algorithm is measured as a function of the length of the input, n, and the amount of disorder. Hence, the running time varies between $O(n)$ time and $O(n \lg n)$ depending on the amount of disorder.
- The algorithm should be adaptive without knowing the amount of disorder beforehand.

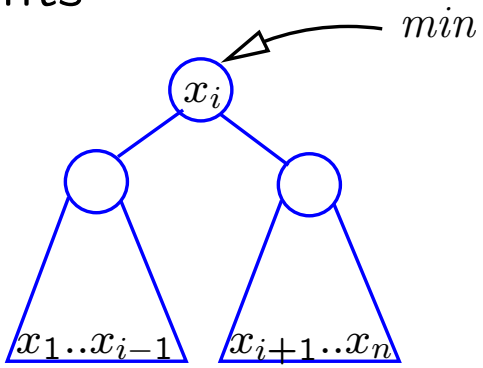
Let $\langle x_1, x_2, \ldots, x_n \rangle$ be a sequence of n elements. For simplicity, assume that all elements are distinct.

$Inv(n) := \left|\left\{(i,j) \mid 1 \leq i < j \leq n \text{ and } x_i > x_j\right\}\right|$ is one measure of disorder

Adaptive Heapsort

input: sequence $\langle x_1, x_2, \ldots, x_n \rangle$ of n elements

```
1  Construct an empty Cartesian tree C
2  hint ← 0
3  for i ∈ {1, 2, ..., n}
4  |   hint ← C.insert(x_i, hint)
5  Construct an empty priority queue Q
6  Q.insert(C.minimum())
7  for j ∈ {1, 2, ..., n}
8  |   x_j ← Q.extract-min()
9  |   Let Y be the set of children x_j has in C
10 |   for each y ∈ Y
11 |   |   Q.insert(y)
```



Idea: Keep $\mathcal{Q}$ small

[Levcopoulos & Petersson 1993]

Theoretical Race

For priority queue $\mathcal{Q}$, element comparisons $\leq \beta n \lg(Inv(n)/n) + O(n)$

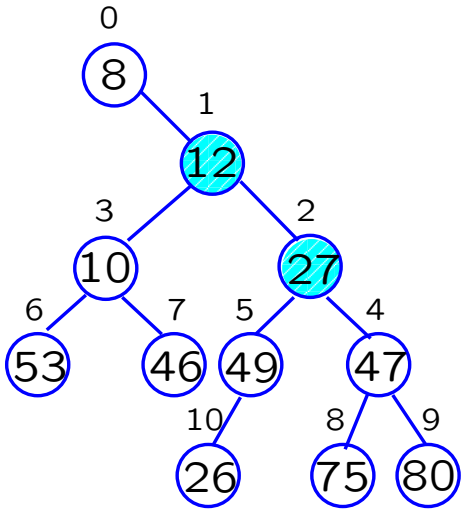
$\mathcal{Q}$	β	Reference
binary heap	3	
combined *extract-min insert*	2.5	[Levcopoulos & Petersson 1993]
binomial queue	2	[folklore]
weak heap	2	
combined *extract-min insert*	1.5	[folklore]
multipartite priority queue	1	[Elmasry, Jensen & Katajainen 2008]

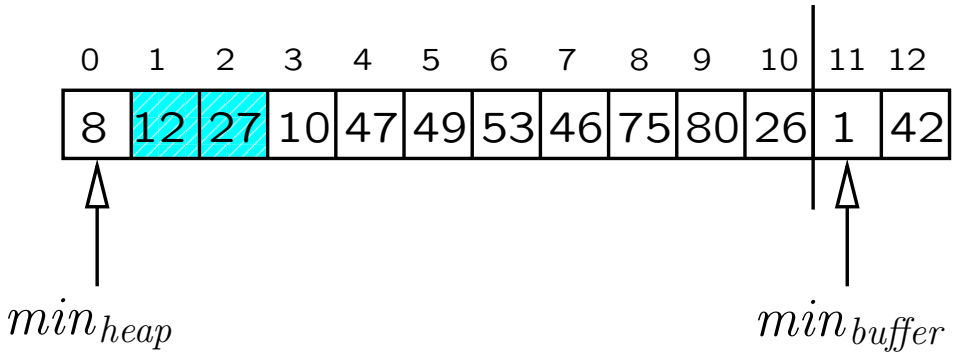
Goal: Achieve the constant-factor optimality, i.e. $\beta = 1$, and in the meantime ensure practicality!

One-Buffer Solution

Task: $insert$ in $O(1)$ amortized time; $extract\text{-}min$ in $O(\lg n)$ worst-case time including at most $\lg n + O(1)$ element comparisons

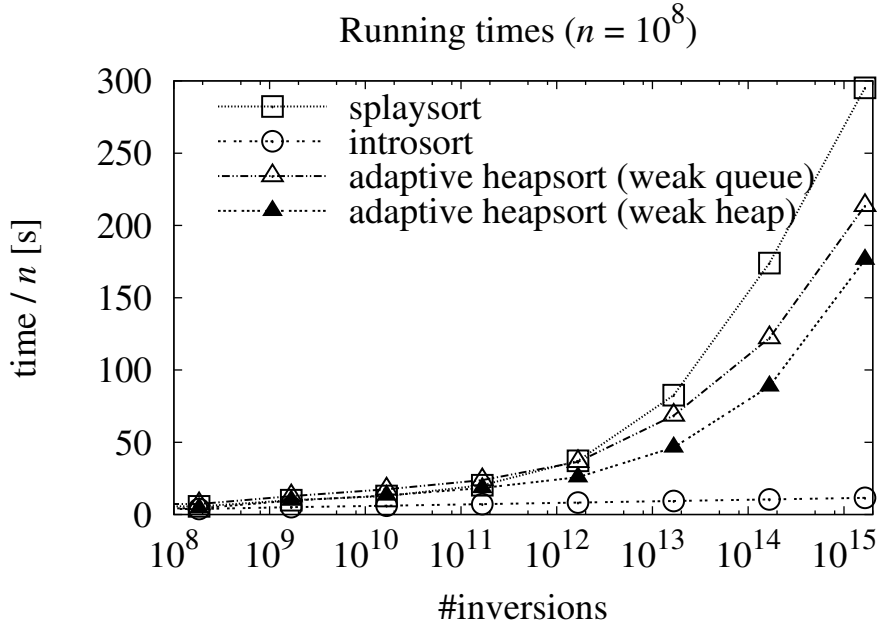
Idea: Temporarily store inserted elements in a buffer and, once it is full (size $\lg n$), move elements to the main structure as a **bulk**

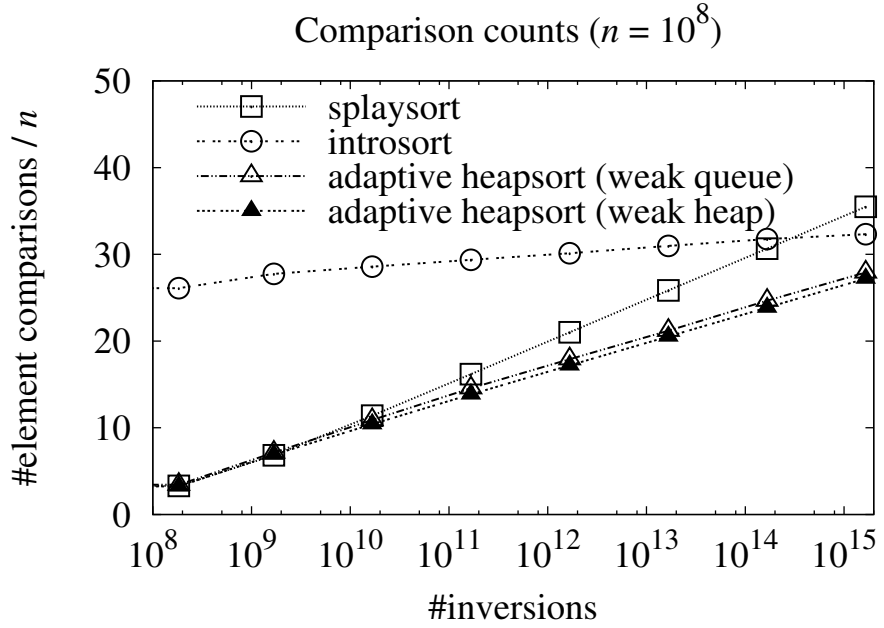




Experiments

Running time and element comparisons when $n = 10^8$





What is the best elementary priority queue?

Best $\sim$ In terms of element comparisons and practical running time

Elementary $\sim$ no handles, no support of $decrease$ and $delete$

Elementary Priority Queues

Lower bounds for binary heaps:
$\Omega(\lg \lg n)$ for $insert$, $\lg n + \log^* n - O(1)$ for $extract\text{-}min$ (under some assumptions)

Bulk insertion in weak heaps:
$O(1)$ amortized for $insert$, $\lg n$ for $extract\text{-}min$

$\rightsquigarrow$ **Engineered weak heaps:**
$O(1)$ for $insert$, $\lg n$ for $extract\text{-}min$, space $n/w + O(1)$

Bulk insertion in binary heaps:
$O(1)$ amortized for $insert$, $\lg n$ amortized for $extract\text{-}min$

$\rightsquigarrow$ **Optimal in-place heaps:**
$O(1)$ for $insert$, $\lg n$ for $extract\text{-}min$, space $O(1)$ [submitted]

Performance of Some Priority Queues

Data structure	Space	*insert*	*extract-min*
binary heaps [Wil64]	$O(1)$	$\lg n + O(1)$	$2 \lg n + O(1)$
binom. queues [Bro78,Vui78]	$O(n)$	$O(1)$	$2 \lg n + O(1)$
heaps on heaps [GM86]	$O(1)$	$\lg \lg n + O(1)$	$\lg n + \log^* n + O(1)$
queue of pennants [CMP88]	$O(1)$	$O(1)$	$3 \lg n + \log^* n + O(1)$
multipartite PQs [EJK08]	$O(n)$	$O(1)$	$\lg n + O(1)$
engin. weak heaps [EEK13]	$n/w + O(1)$	$O(1)$	$\lg n + O(1)$
optimal in-place heaps	$O(1)$	$O(1)$	$\lg n + O(1)$

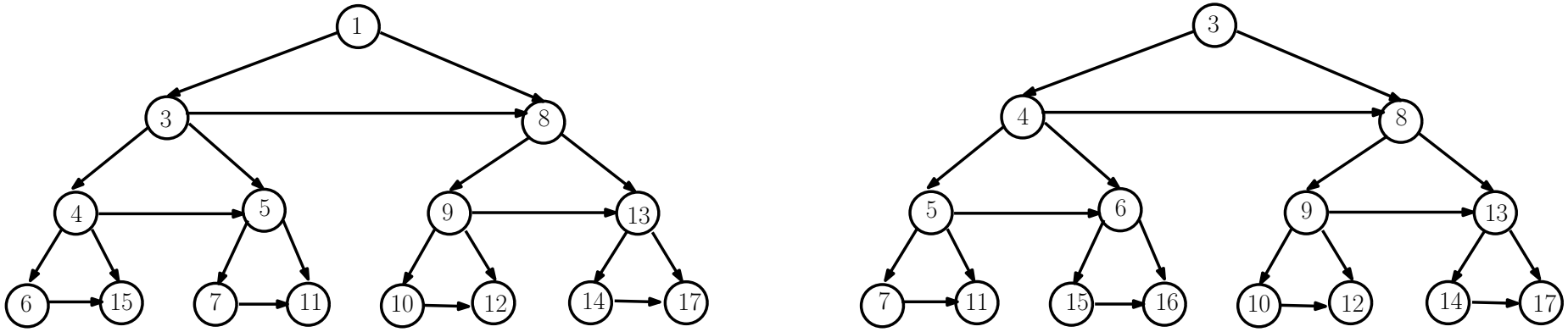
All data structures support *construct* in $O(n)$ and *minimum* in $O(1)$ worst-case time

Strong Heaps

A **strong heap** is a binary heap where nodes dominate their right siblings

Rotating sift-down

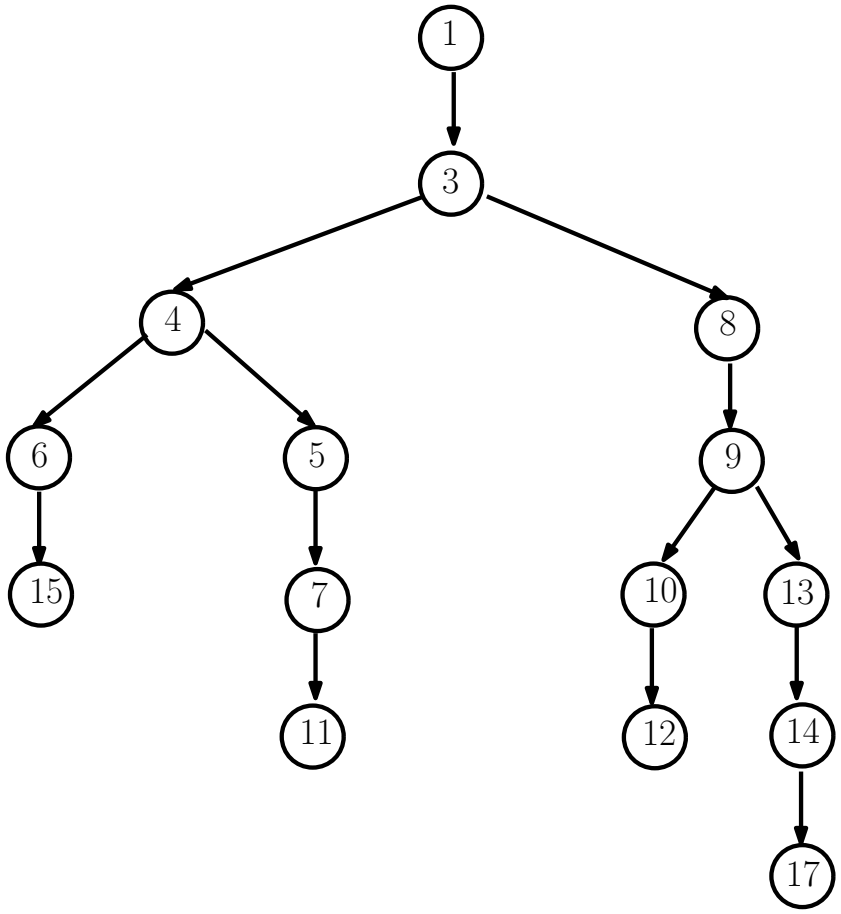
Replace the minimum with 16

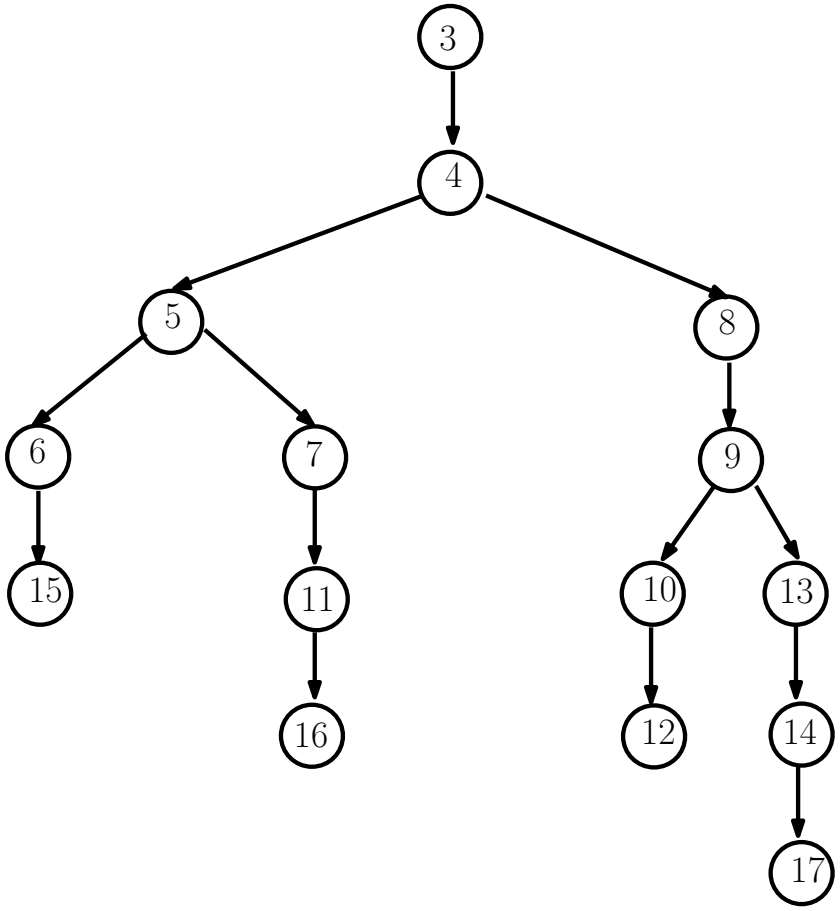


Strong Heaps

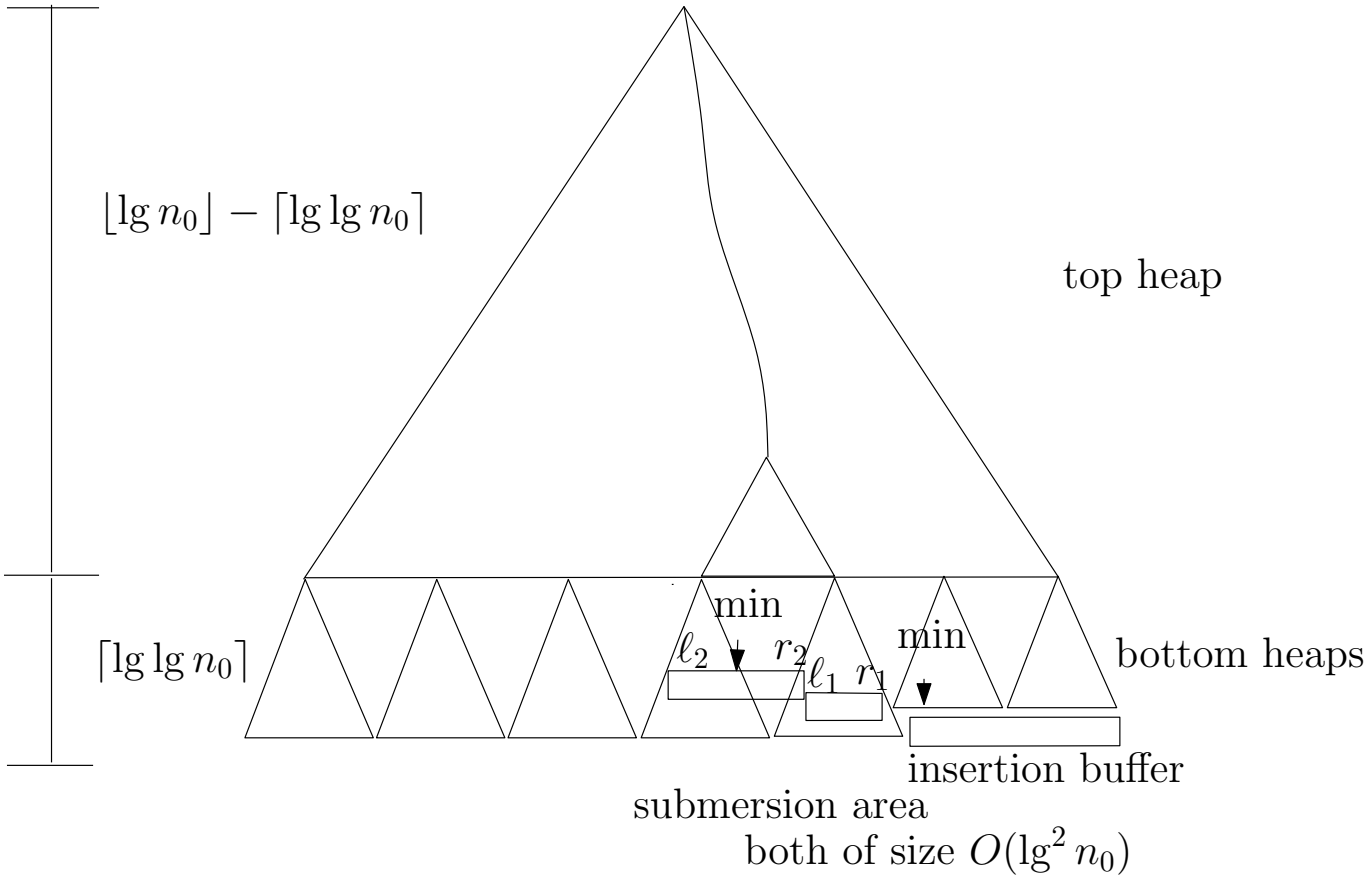
Strong sift-down

Replace the minimum with 16





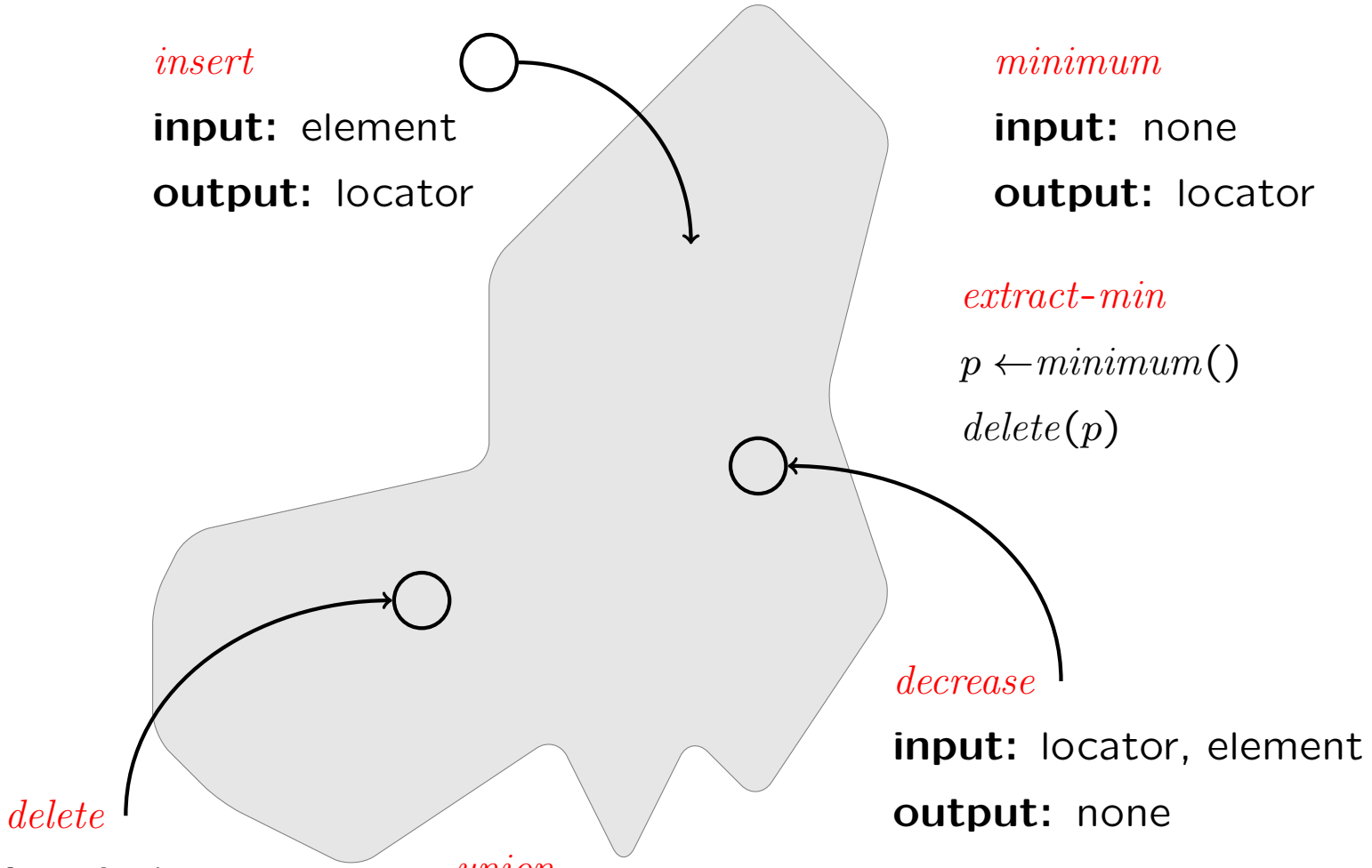
Optimal In-Place Heaps



What is the best bound when handling a request sequence consisting of n *insert*, n *extract-min*, and m *decrease* operations?

Best $\sim$ In terms of element comparisons and practical running time

Addressable Priority Queues



Market Analysis

Efficiency / Operation	binary heap worst case	binomial queue worst case	Fibonacci heap amortized	run-relaxed heap worst case
minimum	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
insert	$\Theta(\lg n)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
decrease	$\Theta(\lg n)$	$\Theta(\lg n)$	$\Theta(1)$	$\Theta(1)$
delete	$\Theta(\lg n)$	$\Theta(\lg n)$	$\Theta(\lg n)$	$\Theta(\lg n)$
union	$\Theta(\lg m \times \lg n)$	$\Theta(\min\{\lg m, \lg n\})$	$\Theta(1)$	$\Theta(\min\{\lg m, \lg n\})$

Here m and n denote the number of elements in the priority queues just prior to the operation.

Result

Rank-relaxed weak heaps are **better** than Fibonacci heaps!

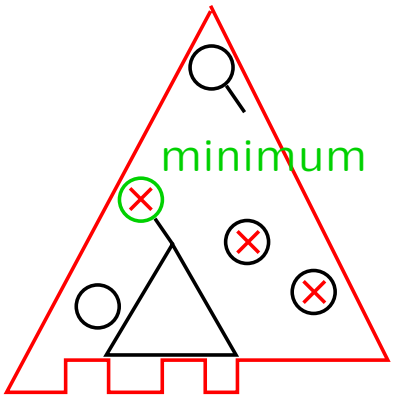
Data structure	Element comparisons
Fibonacci heap	$2m + 2.89n \lg n$
Rank-relaxed weak heap	$2m + 1.5n \lg n$

But they are not **simpler**!

Data structure	Lines of code
Binary heap	205
Fibonacci heap	296
Rank-relaxed weak heap	883

Pointer-Based Representation

- Three pointers per node: left child, right child, parent
- One element per node; in the right subtree no element is larger
- Except the root, the nodes that have at most one child are at the last two levels only
- Recall where the minimum is

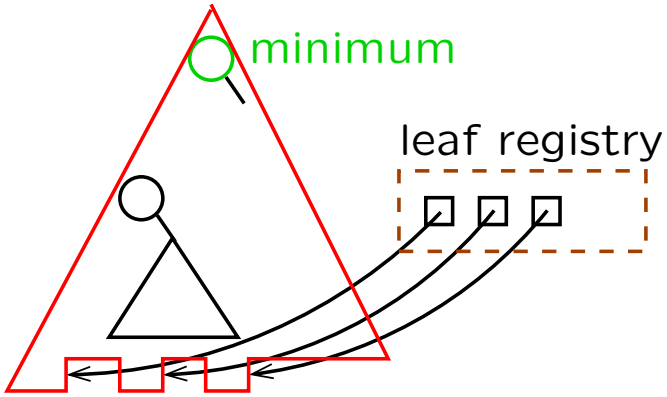


Run relaxation: Remove a potential violation if absolutely necessary (lazy)

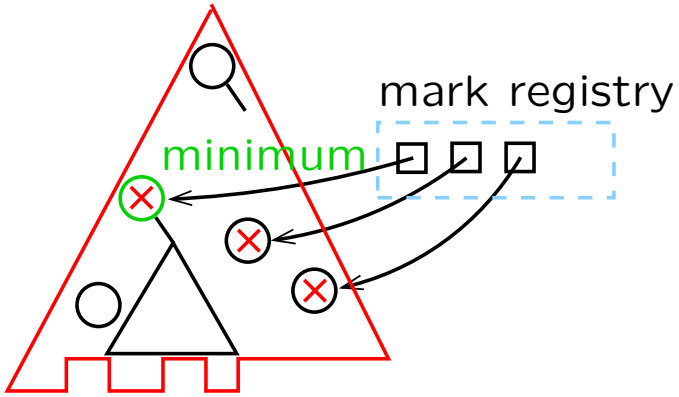
Rank relaxation: Remove a potential violation whenever possible (eager)

Registries

Leaf registry: Keep track of the nodes that have one child or no children

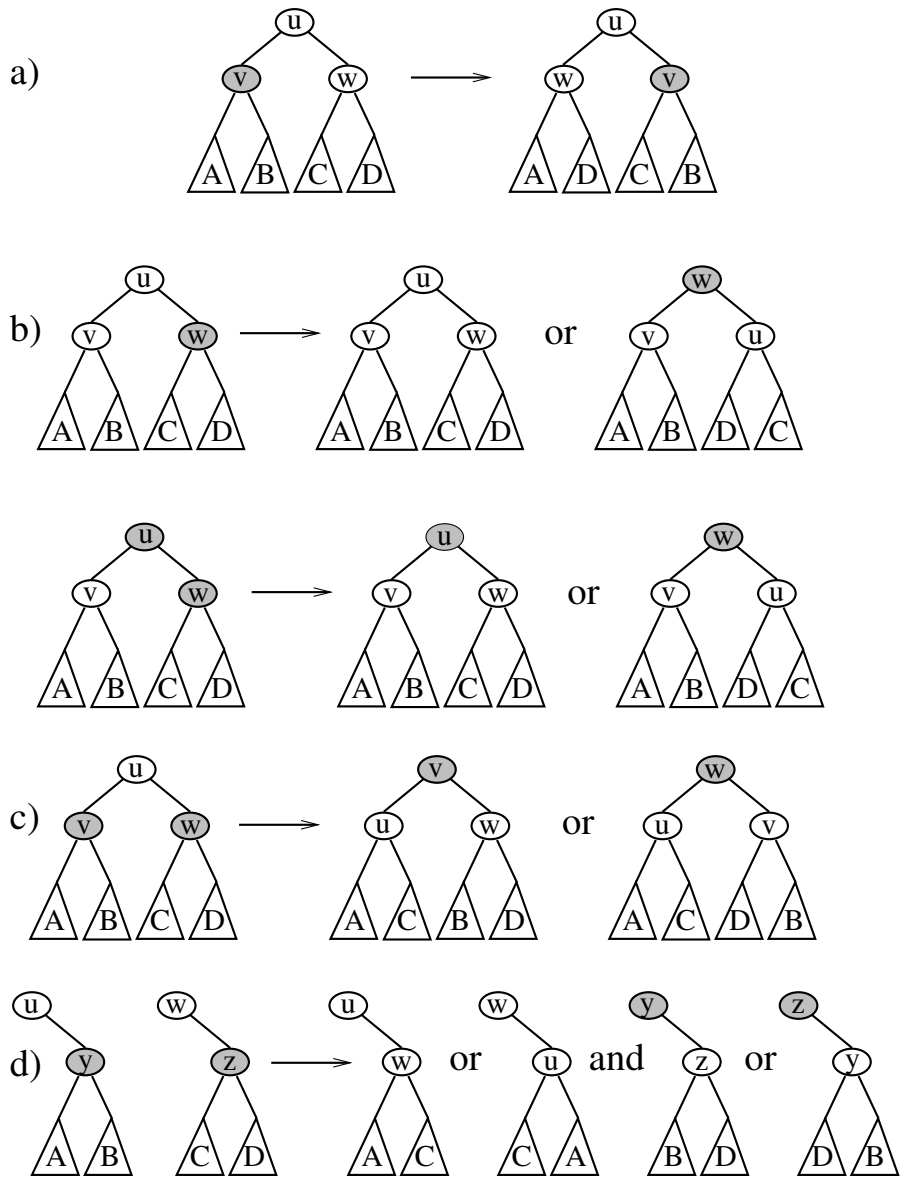


Mark registry: Keep track of the potential violations



Basic operations in both: Location-based $insert$ and $delete$

Transformations



a) cleaning transformation

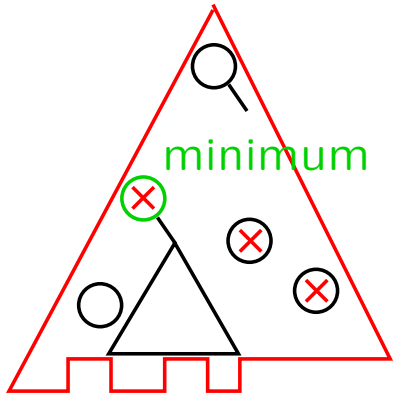
b) parent transformation

c) sibling transformation

d) pair transformation

gray nodes are marked

Rank-Relaxed Weak Heap



- $\lambda \leq \lfloor \lg n \rfloor - 1$ nodes marked; they may violate the weak-heap ordering

insert: Insert a leaf, mark it, apply λ-reducing transformations as long as possible.

decrease: Decrease the value in the given node, mark it, apply λ-reducing transformations as long as possible.

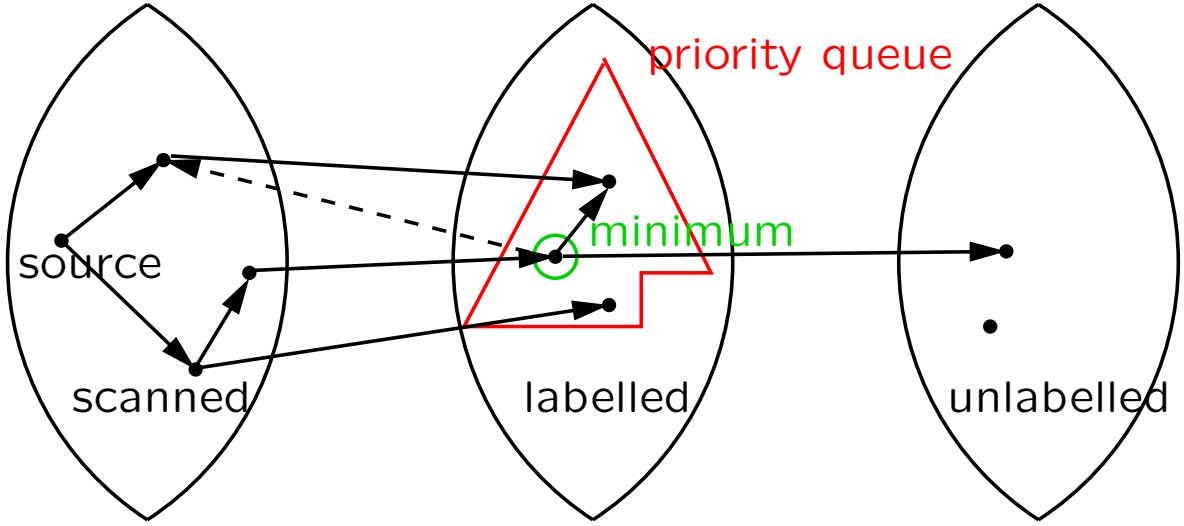
extract-min: Find the minimum (at the root or one of the marked nodes), borrow a leaf, fix the structure of the subtree that lost its root, mark the root of the fixed subtree, apply λ-reducing transformations as long as possible.

Improvement in *extract-min*: If the mark registry is more than half full before the minimum finding, empty it.

Our Play with Dijkstra's Algorithm

With your search engine, you will find many experimental studies on Dijkstra's algorithm. Be critical when you read the results.

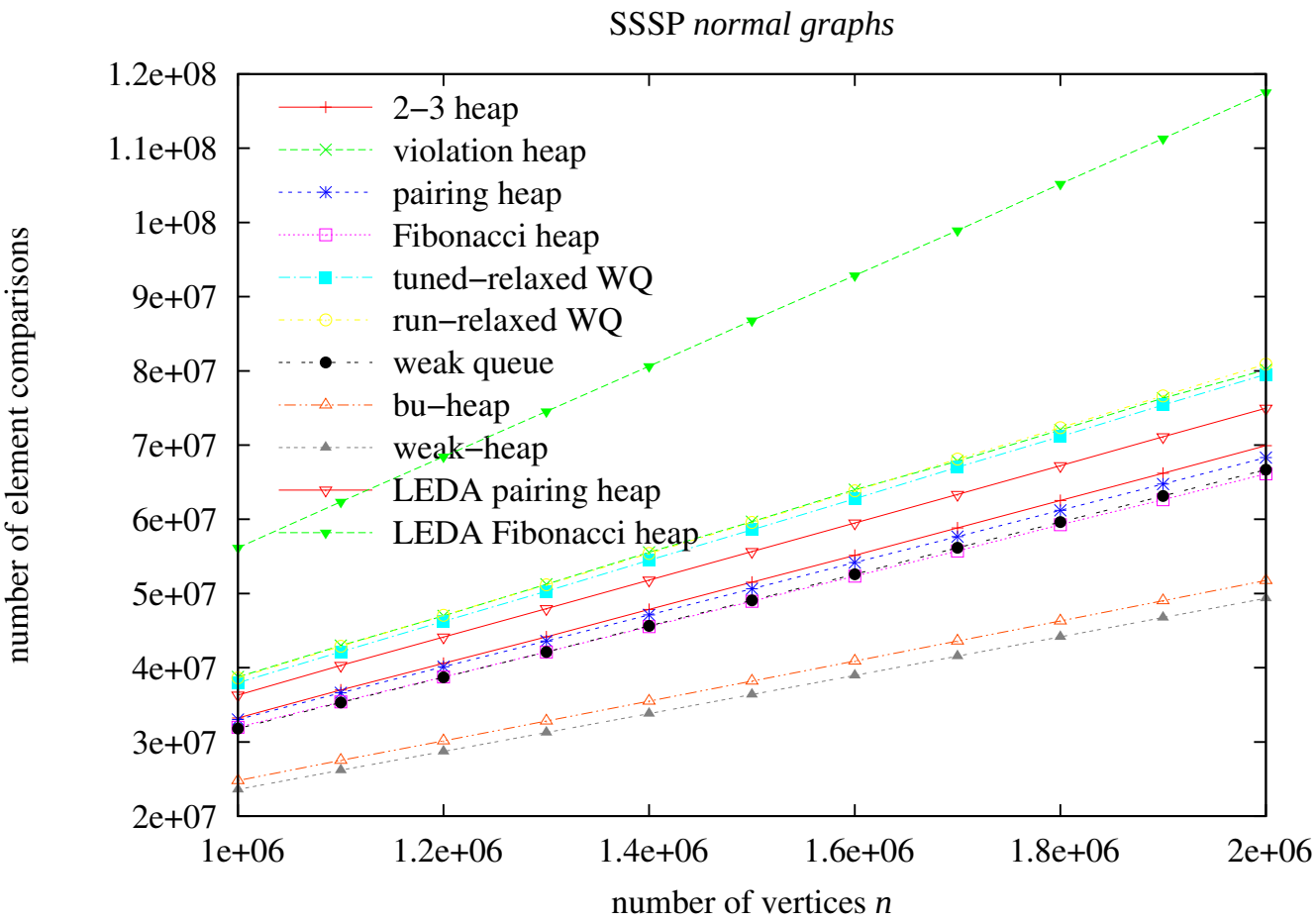
- Which algorithm
- Which graph representation
- Which priority queue
- Which tuning level



a factor of two speed-up

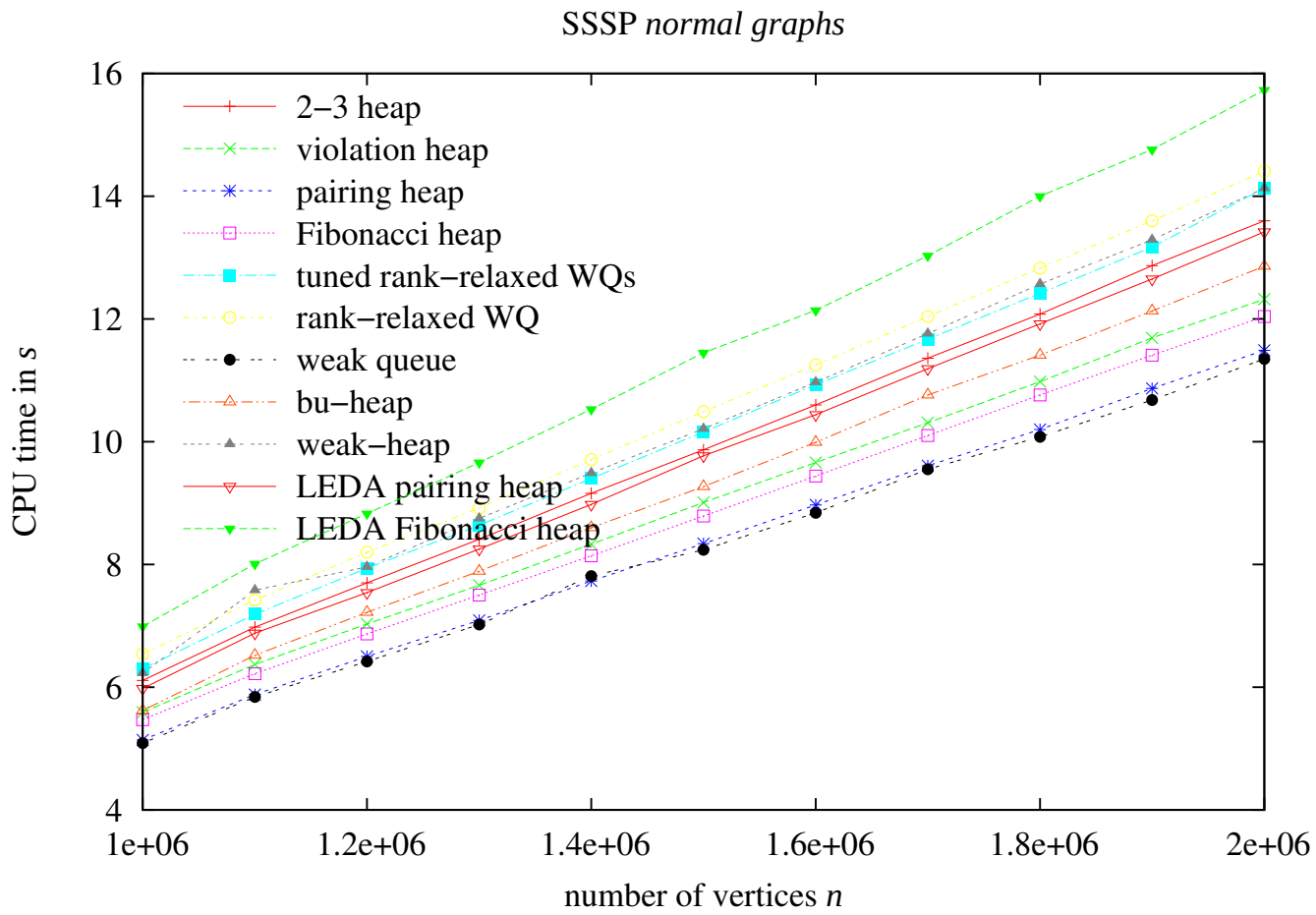
Policy-Based Benchmarking

Element comparisons



Policy-Based Benchmarking

Running time



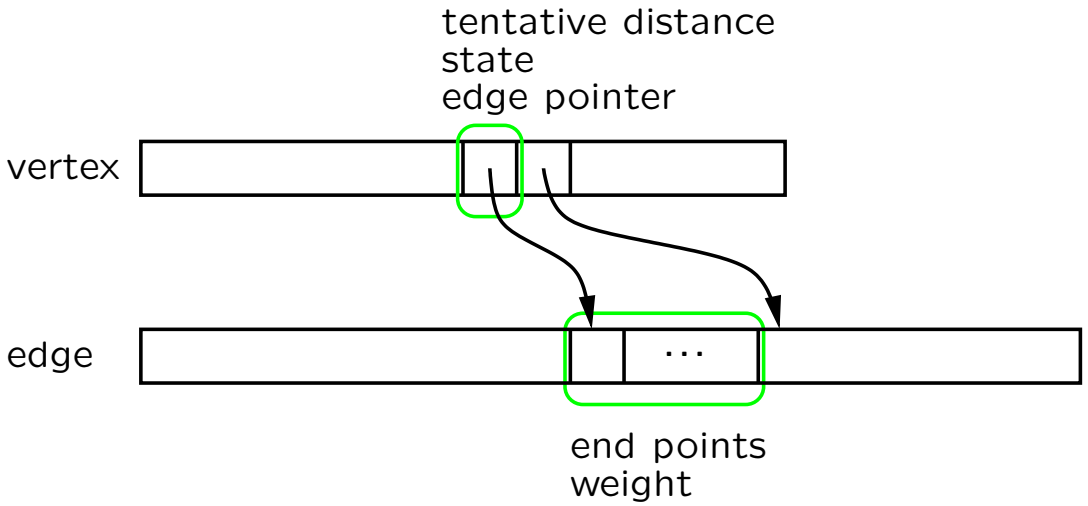
Graph Representation

CPH STL

- adjacency arrays
- simple
- static
- $16m + 16n + O(1)$ bytes for a graph with m edges and n vertices

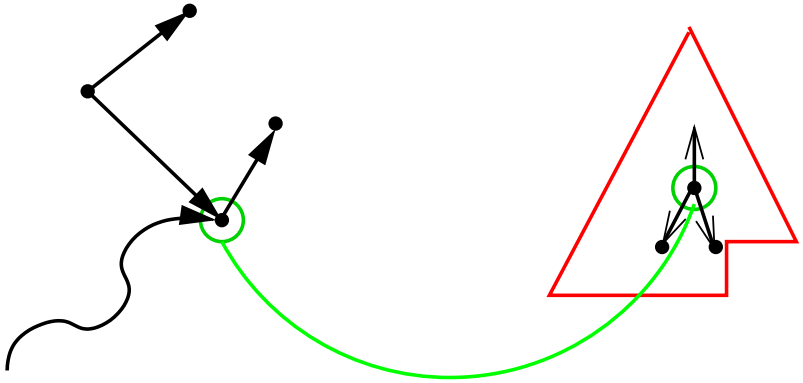
LEDA

- adjacency lists
- nice interface
- fully dynamic
- parameterized
- $52m + 60n + O(1)$ bytes [LEDA Book, § 6.14]



a factor of two speed-up

Avoiding Indirection



Combine the graph vertex and the priority-queue node [Knuth 1994]
$\rightarrow$ improves cache behaviour

a factor of two speed-up

Tuning

Running time per n [μs]

Structure / Operation	CPH STL Fibonacci heap	LEDA 6.2 Fibonacci heap
insert		
n: 10 000	0.10	0.18
n: 100 000	0.09	0.15
n: 1 000 000	0.09	0.15
decrease		
n: 10 000	0.03	0.06
n: 100 000	0.05	0.22
n: 1 000 000	0.06	0.31
extract-min		
n: 10 000	0.7	1.2
n: 100 000	1.4	2.7
n: 1 000 000	2.8	4.5

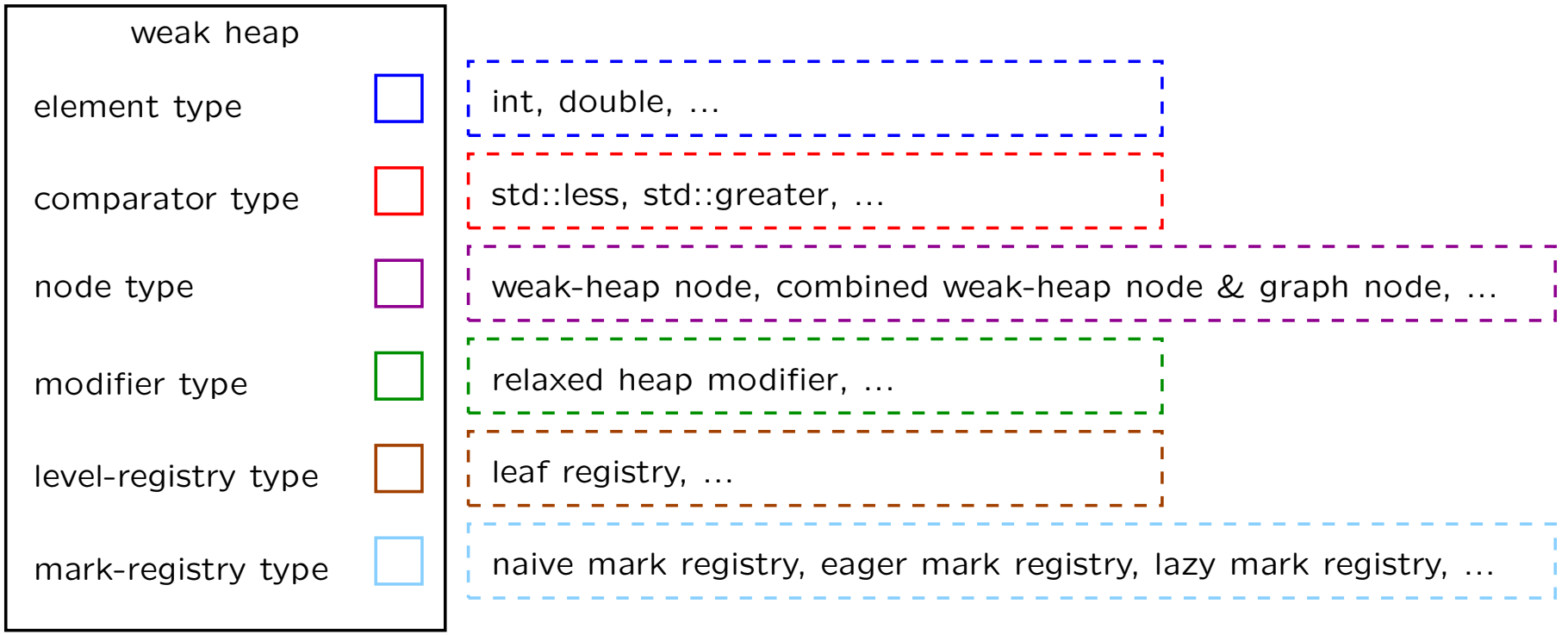
Element comparisons per n

Structure / Operation	CPH STL Fibonacci heap	LEDA 6.2 Fibonacci heap
insert		
n: 10 000	0	1
n: 100 000	0	1
n: 1 000 000	0	1
decrease		
n: 10 000	0	2
n: 100 000	0	2
n: 1 000 000	0	2
extract-min		
n: 10 000	16.2	29.9
n: 100 000	21.2	38.3
n: 1 000 000	26.2	46.5

a factor of two speed-up

On my computer (Ubuntu, g++, with -O3)

Parameterized Design



- comparators shared
- nodes shared
- transformations shared
- level registries shared
- mark registries shared

a factor of two less code

What Is the Best?

Our reference sequence

Theory: rank-relaxed weak heap
Dijkstra—time: binary heap
[Williams 1964]
Dijkstra—comps: weak heap
[Dutton 1993]

Worst case per operation

insert—**time:** Fibonacci heap
[Fredman & Tarjan 1987]
insert—**comps:** Fibonacci heap
decrease—**time:** Fibonacci heap
decrease—**comps:** Fibonacci heap
extract-min—**time:** weak queue
[Vuillemin 1978]
extract-min—**comps:** weak heap

Questions

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